

PREPARED FOR SUBMISSION TO JHEP

Constraints on the second order transport coefficients of an uncharged fluid

Sayantani Bhattacharyya^a

^a*Harish Chandra Research Institute,
Chhatnag Rd, Jhansi, Allahabad.*

E-mail: sayanta@hri.res.in

ABSTRACT: In this note we have tried to determine how the existence of a local entropy current with non-negative divergence constrains the second order transport coefficients of an uncharged fluid, following the procedure described in [1]. Just on symmetry ground the stress tensor of an uncharged fluid can have 15 transport coefficients at second order in derivative expansion. The condition of entropy-increase gives five relations among these 15 coefficients. So finally the relativistic stress tensor of an uncharged fluid can have 10 independent transport coefficients at second order.

Contents

1	Introduction	1
2	Brief Summary of our Procedure	6
3	Classification of fluid data	7
3.1	Independent Data	8
3.2	Composite Expressions	11
4	Entropy current	13
4.1	Entropy current in equilibrium and 1st order correction	13
4.2	General constraints on second and the third order corrections	15
4.3	Implementing the general rules at second order	19
4.4	Implementing the general rules at third order	20
5	Constraints on 2nd order transport coefficients	30
5.1	Derivation of the constraints	30
5.2	Comparison with [2]	33
5.3	Comparison with [1]	34
6	Conformal limit	35
7	Acknowledgement	37
8	Appendices	37
A	Identities	37
B	Computation of the divergence	40

1 Introduction

Fluid dynamics is an effective description of near equilibrium physics. It captures the dynamics of locally equilibrated systems in which the parameters of equilibrium vary slowly compared to relaxation length scale. When, for instance, microscopic dynamics is well described by kinetic theory, the Boltzman equation reduces to the equations of fluid dynamics on length scales that are large compared to the molecular mean free path.

The variables of fluid mechanics are the local values of the parameters that characterize fluid equilibrium; in the simplest context these are the fluid temperature, chemical potentials and velocity. The equations of fluid dynamics are simply the conservation of stress-tensor and all other charged currents, once the stress tensor and currents are expressed in terms of equilibration parameters. The formulas that express the stress tensor and charge currents as functions of fluid variables are known as constitutive relations. As fluid dynamics is a long wavelength effective description, it is meaningful only to present constitutive relations in an expansion in derivatives of the fluid variables.

For any given fluid, a microscopic computation of constitutive relations starting from a microscopic description of the system is often an impossibly difficult task. In the usual spirit of effective field theory, this task is, moreover, extraneous to the study of fluid dynamics. An autonomous ‘theory’ of fluid dynamics addresses the following question: what is the most general form of the constitutive relations that could possibly arise in the fluid description of any consistent system (to any given order in the derivative expansion).

The requirement of symmetry restricts the form of the constitutive relations. For example, the stress tensor is a tensor. At any given order in the derivative expansion, there exists only a finite number of (onshell inequivalent) tensor structures one can build out of fluid variables and their derivatives. The most general expression for the stress tensor is clearly given by a linear combination of these inequivalent tensors, where the coefficients in this expansion are arbitrary functions of the scalar fluid variables (temperatures and chemical potentials in the simplest situations). While the requirements of symmetry are certainly necessary, they are not sufficient. There is atleast one additional constraint on allowed constitutive relations: that they are consistent with a local form of the second law of thermodynamics [3]. In other words any given constitutive relation must be accompanied by an entropy current, also constructed out of fluid variables. The entropy current must have the property that its divergence is positive for every conceivable fluid flow allowed by the fluid equations with the specified constitutive relations ¹.

It is a quite remarkable fact that the requirement of the existence of a positive divergence entropy current constrains the allowed constitutive relations of fluid dynamics in a quite dramatic manner; as we will see in some detail below, this requirement reduces the number of free parameters (or more precisely free functions) allowed in constitutive relations.

In this note we will work the restrictions imposed by this requirement on the constitutive relations of an uncharged relativistic fluid in $3+1$ spacetime dimensions at second order in the derivative expansion. In this system the variables of fluid

¹However, the existence of a local entropy current with positive divergence is really a heuristic, with as yet no solid basis in thermodynamics or QFT

dynamics are simply the temperature T and the fluid four velocity u^μ . As we will see in some detail below, symmetry considerations allow a 15 parameter worth of constitutive relations for the stress tensor, where every parameter is an arbitrary function of the temperature. It was already noted by Romatschke [1] that the requirement of entropy increase imposes atleast two relations between these 15 functions. In this note we generalize Romatschke's analysis and demonstrate that a complete study of the requirements of positivity of the entropy current imposes 5 relations on the 15 coefficients described above. The set of all 2nd order constitutive relations consistent with the positivity of entropy increase is parameterized by ten functions of the temperature. But unlike the first order case we did not find any inequalities among the second order transport coefficients.

We now give a detailed presentation of our final results, i.e. a precise characterization of the 10 parameter set of allowed constitutive relations at second order in the derivative expansion for uncharged relativistic fluids. Let the fluid stress tensor be given by

$$T_{\mu\nu} = T_{\mu\nu}^{perf} + \Pi^{\mu\nu}$$

We work in the so called Landau frame which imposes the transversality condition

$$u^\mu \Pi_{\mu\nu} = 0$$

In this frame the most general allowed form for $\Pi_{\mu\nu}$ upto second order in the derivative expansion is given by²

$$\begin{aligned} \Pi_{\mu\nu} = & -\eta\sigma_{\mu\nu} - \zeta P_{\mu\nu}\Theta \\ & T \left[\tau (u \cdot \nabla) \sigma_{\langle\mu\nu\rangle} + \kappa_1 R_{\langle\mu\nu\rangle} + \kappa_2 F_{\langle\mu\nu\rangle} + \lambda_0 \Theta \sigma_{\mu\nu} \right. \\ & \left. + \lambda_1 \sigma_{\langle\mu}{}^a \sigma_{a\nu\rangle} + \lambda_2 \sigma_{\langle\mu}{}^a \omega_{a\nu\rangle} + \lambda_3 \omega_{\langle\mu}{}^a \omega_{a\nu\rangle} + \lambda_4 \mathbf{a}_{\langle\mu} \mathbf{a}_{\nu\rangle} \right] \\ & + TP_{\mu\nu} \left[\zeta_1 (u \cdot \nabla) \Theta + \zeta_2 R + \zeta_3 R_{00} + \xi_1 \Theta^2 + \xi_2 \sigma^2 + \xi_3 \omega^2 + \xi_4 \mathbf{a}^2 \right] \end{aligned} \quad (1.1)$$

²Our convention for Riemann tensor is the following

$$R^\rho{}_{\alpha\beta\nu} = \partial_\beta \Gamma^\rho_{\alpha\nu} - \partial_\nu \Gamma^\rho_{\alpha\beta} + \Gamma^\lambda_{\alpha\nu} \Gamma^\rho_{\lambda\beta} - \Gamma^\lambda_{\alpha\beta} \Gamma^\rho_{\lambda\nu}$$

where

$$\begin{aligned}
u^\mu &= \text{The normalised four velocity of the fluid} \\
P^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu = \text{Projector perpendicular to } u^\mu \\
\Theta &= \nabla \cdot u = \text{Expansion, } \mathbf{a}_\mu = (u \cdot \nabla) u_\mu = \text{Acceleration} \\
\sigma^{\mu\nu} &= P^{\mu\alpha} P^{\nu\beta} \left(\frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha}{2} - \frac{\Theta}{3} g_{\alpha\beta} \right) = \text{Shear tensor} \\
\omega^{\mu\nu} &= P^{\mu\alpha} P^{\nu\beta} \left(\frac{\nabla_\alpha u_\beta - \nabla_\beta u_\alpha}{2} \right) = \text{Vorticity} \\
F^{\mu\nu} &= R^{\mu\alpha\nu b} u_a u_b, \quad R^{\mu\nu} = R^{a\mu b\nu} g_{ab} \quad (R^{abcd} = \text{Reimann tensor}) \\
\sigma^2 &= \sigma_{\mu\nu} \sigma^{\mu\nu}, \quad \omega^2 = \omega_{\mu\nu} \omega^{\mu\nu}
\end{aligned} \tag{1.2}$$

and

$$A_{\langle\mu\nu\rangle} \equiv P_\mu^\alpha P_\nu^\beta \left(\frac{A_{\alpha\beta} + A_{\beta\alpha}}{2} - \left[\frac{A_{ab} P^{ab}}{3} \right] g_{\alpha\beta} \right) \quad \text{For any tensor } A_{\mu\nu}$$

It turns out that ‘entropy-positivity’ does not impose any constraint on τ , λ_0 , λ_1 , λ_2 , ζ_1 , ξ_1 and ξ_2 . The rest of the eight second order transport coefficients satisfy the following 5 relations.

$$\begin{aligned}
\kappa_2 &= \kappa_1 + T \frac{d\kappa_1}{dT} \\
\zeta_2 &= \frac{1}{2} \left[s \frac{d\kappa_1}{ds} - \frac{\kappa_1}{3} \right] \\
\zeta_3 &= \left(s \frac{d\kappa_1}{ds} + \frac{\kappa_1}{3} \right) + \left(s \frac{d\kappa_2}{ds} - \frac{2\kappa_2}{3} \right) + \frac{s}{T} \left(\frac{dT}{ds} \right) \lambda_4 \\
\xi_3 &= \frac{3}{4} \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \left(T \frac{d\kappa_2}{dT} + 2\kappa_2 \right) - \frac{3\kappa_2}{4} + \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \lambda_4 \\
&\quad + \frac{1}{4} \left[s \frac{d\lambda_3}{ds} + \frac{\lambda_3}{3} - 2 \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \lambda_3 \right] \\
\xi_4 &= -\frac{\lambda_4}{6} - \frac{s}{T} \left(\frac{dT}{ds} \right) \left(\lambda_4 + \frac{T}{2} \frac{d\lambda_4}{dT} \right) - T \left(\frac{d\kappa_2}{dT} \right) \left(\frac{3s}{2T} \frac{dT}{ds} - \frac{1}{2} \right) \\
&\quad - \frac{Ts}{2} \left(\frac{dT}{ds} \right) \left(\frac{d^2\kappa_2}{dT^2} \right)
\end{aligned} \tag{1.3}$$

So finally there are 10 independent transport coefficients at second order for some uncharged fluid.

As we have explained above, unless the equations (1.3) are satisfied, the fluid dynamics equations do not have a positive divergence entropy current. When the equations (1.3) are satisfied the fluid equations are compatible with the existence of a positive divergence entropy current, but this current is not unique. Consider a fluid with a particular constitutive relation that obeys the equations (1.3). It turns

out that any such fluid has a 7 parameter (7 arbitrary functions of temperature) set of positive divergence entropy currents.

$$\begin{aligned}
& J^\mu|_{\text{upto 2nd order}} \\
&= su^\mu + \nabla_\nu [A_1(u^\mu \nabla^\nu T - u^\nu \nabla^\mu T)] + \nabla_\nu (A_2 T \omega^{\mu\nu}) \\
&+ A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu + \left(\frac{A_3}{T} + \frac{dA_3}{dT} \right) [\Theta \nabla^\mu T - P^{ab} (\nabla_b u^\mu) (\nabla_a T)] \\
&+ (B_1 \omega^2 + B_2 \Theta^2 + B_3 \sigma^2) u^\mu + B_4 [(\nabla s)^2 u^\mu + 2s \Theta \nabla^\mu s]
\end{aligned} \tag{1.4}$$

Our results above apply to an arbitrary theory, i.e. a theory with an arbitrary equation of state. The specialization of our results to the special case of conformal fluids turns out to be trivial, as we now explain. As was explained in ([4, 5]), the requirement of Weyl invariance forces 10 linear combinations of the 15 symmetry allowed coefficients to vanish; specifically

$$\begin{aligned}
\kappa_1 = 2\kappa_2 \equiv \kappa, \quad \tau = 3\lambda_0, \quad \lambda_4 = 0 \\
\zeta_1 = \zeta_2 = \xi_1 = \xi_2 = \xi_3 = 0
\end{aligned} \tag{1.5}$$

Moreover, the the temperature dependence of the remaining five coefficients (those that are allowed to take arbitrary values consistent with Weyl invariance) is determined by dimensional analysis; All of them are just linearly proportional to temperature.

It turns out that this linear dependence on temperature, the conformal equation of state and (1.5) reduce all of the equations (1.3) to trivial identities of the form $0 = 0$. In other words, the requirement of positivity invariance of the entropy current does not impose any equations on the five transport coefficients (allowed by conformal symmetry).

The coefficients for a conformally covariant entropy current are given by the following expressions.

$$\begin{aligned}
A_1(T) = a_1, \quad A_2(T) = a_2, \quad A_3(T) = \frac{a_1}{2} T \\
B_1(T) = b_1 T, \quad B_2(T) = \frac{2a_1}{9} T, \quad B_3(T) = b_3 T \\
B_4(T) = - \left(\frac{a_1}{18} \right) T^{-5}
\end{aligned} \tag{1.6}$$

where all a_i and b_i are constants.

Therefore the conformally covariant entropy current has four independent coefficients (a_1 , a_2 , b_1 and b_3) when expanded upto second order in derivatives.

Let us end this introduction with a description of our motivations in undertaking the computations described in this note. Our first motivation is practical. Theoretical reconstructions of the RHIC and LHC heavy ion experiments often model the

expansion of the hot dense deconfined plasma by the equations of fluid dynamics including second order corrections. Given that we have not been able, from first principles, to compute the fluid description of QCD it seems of interest to parameterize the most general set of allowed equations, as we have done (to 2nd order) in this note.

However our main motivation in undertaking the computations described in this note are structural. At zero temperature, the equations of motion of physical systems are strongly constrained by the requirement that they follow from the extremization of an action. On the other hand the radiational formulation of the equations of fluid dynamics is at the level of the equations of motion. It seems likely that the equations of fluid dynamics inherit constraints that are the analogue of the zero temperature requirement of being obtained from an action. One possible source of such constraints, as described in this note, stems from the requirement that our system admit an entropy current of positive divergence. There are other potential sources of constraints, for instance the requirement that correlation functions computed from fluid dynamics have certain symmetry properties that can be derived, on general grounds, in quantum field theories (see [6]). It appears to us to be of interest to find a complete ‘theory’ of fluid dynamics; a formalism that enumerates all consistency conditions on the equations of fluid dynamics. Such a formalism would take the place of the zero temperature requirement that the equations of motion follow from an action. The computations presented in this note may be thought of as a small first step towards this larger goal.

2 Brief Summary of our Procedure

In the rest of this note we proceed to determine the most general second order constitutive relations and second order entropy current consistent with positivity of the divergence of the current. In order to do this we first list out the most general symmetry allowed entropy current upto third order in the derivative expansion (onshell equivalent currents are not treated as distinct). We then perform a brute force computation of the divergence of this entropy current, keeping all relevant terms (see below for an explanation of which terms are relevant) to fourth order in the derivative expansion. We then use the equations of motion (including constitutive terms upto second order in derivatives) to rewrite our final answer entirely as a function of derivatives of velocity and temperature and the background metric that are independent of each other (i.e. are not related to each other by the equations of motion). We then work out the conditions on the entropy current and constitutive relations that ensure the positivity of this divergence for arbitrary values of the independent fluid derivatives, finally obtaining (1.3) and (1.4)

In order to implement the programme outlined in the paragraph above, in this note we proceed in the following order. In section 3 below we classify and enumerate

the onshell independent derivatives of fluid fields (upto fourth order in derivatives). We also enumerate all the products of these derivative fields with net derivative number ≤ 4 , organizing our enumeration in representations of the local $SO(3)$ that leaves the fluid velocity fixed.

In section 4 below we then proceed to enumerate the most general entropy current upto third order in the derivative expansion. We then compute the divergence of this entropy current and determine several constraints on the entropy current that follow from the requirement that its divergence is positive definite.

In section 5 below we then enumerate the constraints on constitutive relations, upto second order, that follow from the requirement of positivity of divergence of the entropy current.

We end this brief section by listing our conventions. Throughout this note we work in the Landau gauge. In this gauge the velocity u^μ at any point is defined as the unique time-like eigenvector of the stress tensor, normalised so that $u_\mu u^\mu = -1$. In other words, by definition

$$T^\nu_\mu u_\nu = -\epsilon u_\mu \quad (2.1)$$

The quantity ϵ is taken by definition to be energy density of our fluid. All other fluid thermodynamical quantities (like the temperature or pressure) are obtained from ϵ using thermodynamics and equation of state. Equation of state expresses the energy density ϵ as a function of some thermodynamic parameter like entropy density and it can vary from system to system. In this note we shall keep it arbitrary. Once $\epsilon(s)$ is known, the temperature (T) and the pressure (P) can be determined in the following way.

$$T(s) = \frac{d\epsilon(s)}{ds}, \quad P(s) = s \frac{d\epsilon(s)}{ds} - \epsilon(s)$$

Both of the above relations directly follow from equilibrium thermodynamics.

3 Classification of fluid data

³ In this section we present a partial listing of the onshell independent derivatives of fluid fields (T and u^μ), at any given point x , upto fourth order in the derivative expansion. We organize these derivatives (which we will often refer to below as independent data) according to their transformation properties under the $SO(3)$ rotational group that leaves $u^\mu(x)$ invariant.

In order to explain what we mean let us consider a listing of independent data at first order in the derivative expansion. Before accounting for onshell equivalences we have 16 independent pieces of first derivative data; (the four derivatives of temperature and the four derivatives of each of the three independent velocities). These 16 pieces of data transform, under the local $SO(3)$, as two scalars, two vectors, a

³This section has been worked out in collaboration with Shiraz Minwalla and Tarun Sharma.

pseudo vector and a traceless symmetric tensor (i.e. the **5**) of $SO(3)$ (see the second column of Table 1 for details). However these 16 pieces of data are not all independent. The four *perfect* fluid equations of motion may be used to solve for four of these fluid derivatives in terms of the other 12. As the four fluid equations can be decomposed into a vector and a scalar of $SO(3)$ (see the third column of Table 1) it follows that the independent data consists of one vector, one scalar, a pseudo vector and a traceless symmetric tensor (see the fourth column of Table 1). The choice of the independent scalar and vector piece of data is arbitrary; we could take our independent data to be either of the vectors and either of the scalars listed in the second column of Table 1. In the fourth column of Table 1 we have made one particular choice of the independent data that we will employ in much of our note. Occasionally we will find it more convenient to use another choice of independent data; we will explicitly point this out when this is the case.

In this note we will require that the production of entropy is positive for an arbitrary fluid flow on an arbitrary curved manifold. As explained in [7] this requirement yields constraints for the constitutive relations of fluids even in the flat space. In order to implement the constraint described above, we will find it necessary to list the data associated with local background metric curvatures in addition to the data associated with fluid flows. All curvature invariants formed from a background metric are given in terms of (contractions of) the Riemann tensor and its derivatives. It is important to recall that, in addition to certain symmetry properties, the Riemann tensor also obeys a Bianchi type identity. The independent derivatives of the Riemann tensor should be counted modulo the Bianchi identity and its derivatives. In analogy with the counting problem for fluid data listed above, we will regard the set of all derivatives of the Riemann tensor (with all symmetries imposed) as raw data, and the Bianchi identities and its derivatives as ‘equations of motion’. We will then list the independent pieces of data in curvature and derivatives by subtracting equations of motion from raw data, just as described in the previous paragraph.

3.1 Independent Data

With no further ado we simply proceed to list the (relevant parts of) fluid and curvature data at various orders in the derivative expansion.

At first order in derivatives we have only fluid data. They are listed below in Table 1.

Here for any tensor $A_{\mu\nu}$, the symbol $A_{\langle\mu\nu\rangle}$ means the symmetric traceless part of it, projected in the direction perpendicular to u^μ .

$$A_{\langle\mu\nu\rangle} \equiv P_\mu^a P_\nu^b \left[\left(\frac{A_{ab} + A_{ba}}{2} \right) - g_{ab} \left(\frac{P^{\alpha\beta} A_{\alpha\beta}}{3} \right) \right]$$

For example, if we expand this notation, the shear tensor $\sigma_{\mu\nu}$ has the following

Table 1. Data at 1st order in derivative

	Before imposing eom	Eoms	Independent data
Scalars (1)	$(u.\nabla)T, (\nabla.u)$	$u_\nu \nabla_\mu T^{\mu\nu} = 0$	$\Theta \equiv (\nabla.u)$
Vectors (1)	$(u.\nabla)u^\mu, P^{\mu\nu} \nabla_\nu T$	$P_a^\mu \nabla_\nu T^{\nu a} = 0$	$\mathfrak{a}^\mu \equiv (u.\nabla)u^\mu$
Pseudo-vectors (1)	$u_\nu \epsilon^{\nu\mu\lambda\sigma} \nabla_\lambda u_\sigma$		$l^\mu \equiv u_\nu \epsilon^{\nu\mu\lambda\sigma} \nabla_\lambda u_\sigma$
Tensors (1)	$\nabla_{\langle\mu} u_{\nu\rangle}$		$\sigma_{\mu\nu} \equiv \nabla_{\langle\mu} u_{\nu\rangle}$

definition

$$\sigma_{\mu\nu} \equiv P_\mu^a P_\nu^b \left[\frac{\nabla_a u_b + \nabla_b u_a}{2} - g_{ab} \frac{\Theta}{3} \right]$$

At second order we have curvature data along with fluid data. The curvature data is given by the 20 independent components of the Reimannn curvature subject to the identities

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha} \\ R_{\mu\nu\alpha\beta} &= R_{\alpha\beta\mu\nu} \\ R_{\mu[\nu\alpha\beta]} &= 0 \end{aligned}$$

The 20 independent components may be decomposed into $SO(3)$ representations as in Table 2

Table 2. I_2 type curvature data

Scalars (2)	$R \equiv R^{\mu\nu}{}_{\mu\nu},$ $R_{00} \equiv u^\mu u^\nu R_{\mu\nu}$ $\equiv u^\mu u^\nu R^\alpha{}_{\mu\alpha\nu}$
Vectors(1)	$P^{\mu a} R_{ab} u^b$
Tensors(2)	$R_{\langle\mu\nu\rangle}$ $F_{\langle\mu\nu\rangle}$ where $F_{\mu\nu} \equiv u^\alpha u^\beta R_{\mu\alpha\nu\beta}$
Pseudo-tensor	$u_b R_{\langle\mu}{}^{bcd}{}_{\nu\rangle} \epsilon_{\nu\rangle cdq} u^q$

Second order fluid data is tabulated in Table 3

Third order fluid data that transforms in the scalar, vector and pseudo vector representations is tabulated in Table 4 (we will never need 3rd order data in the **5**, **7** and **9** representations, and so do not bother to tabulate these below).

Table 3. I_2 type fluid data

	Before imposing eom	Eoms	Independent data
Scalars (1)	$(u.\nabla)\Theta, \nabla^2 T$ $u^\mu u^\nu \nabla_\mu \nabla_\nu T$	$u_\nu (u.\nabla) \nabla_\mu T^{\mu\nu} = 0,$ $\nabla_\mu \nabla_\nu T^{\mu\nu} = 0$	$(u.\nabla)\Theta$
Vectors (2)	$P^{\mu\nu}(u.\nabla)u_\nu, P^{\mu\nu}\nabla^2 u_\nu,$ $P^{\mu\nu}\nabla_\nu \Theta, P^{\mu\nu}\nabla_\nu (u.\nabla)T$	$P_a^\mu (u.\nabla) \nabla_\nu T^{\nu a} = 0,$ $u_a P^{\mu b} \nabla_b \nabla_\nu T^{\nu a} = 0$	$P^{\mu a} \nabla_a \Theta,$ $P_a^\mu \nabla_b \sigma^{ab}$
Pseudo-vectors (0)	$(u.\nabla)l^\mu$	$u_\nu \epsilon^{\mu\nu\lambda\sigma} \nabla_\lambda \nabla_a T_\sigma^a = 0$	
Tensors (1)	$P^{\mu a} P^{\nu b} (u.\nabla) \sigma_{ab}, \nabla_{\langle\mu} \nabla_{\nu\rangle} T$	$\nabla_{\langle\mu} \nabla_a T_{\nu\rangle}^a = 0$	$P^{\mu a} P^{\nu b} (u.\nabla) \sigma_{ab}$
Pseudo-tensors (1)	$\nabla_{\langle\mu} l_{\nu\rangle}$		$\nabla_{\langle\mu} l_{\nu\rangle}$
Spin-3 (1)	$\nabla_{\langle\mu} \nabla_\nu u_{\alpha\rangle}$		$\nabla_{\langle\mu} \nabla_\nu u_{\alpha\rangle}$

Table 4. I_3 type fluid data

	Before imposing eom	Eoms	Independent data
Scalars (1)	$(u.\nabla)^2 \Theta, \nabla^2 \Theta,$ $(u.\nabla)^3 T, (u.\nabla) \nabla^2 T$	$u_\nu (u.\nabla)^2 \nabla_\mu T^{\mu\nu} = 0,$ $(u.\nabla) \nabla_\mu \nabla_\nu T^{\mu\nu} = 0,$ $u_\nu \nabla^2 \nabla_\mu T^{\mu\nu} = 0$	$(u.\nabla)^2 \Theta$
Vectors (1)	$P^{\mu a} (u.\nabla)^3 u_a$ $P^{\mu a} (u.\nabla) \nabla_a \Theta$ $P^{\mu a} \nabla^2 \nabla_a T$ $P^{\mu a} (u.\nabla)^2 \nabla_a T$ $P^{\mu a} (u.\nabla) \nabla^2 u_a$	$P^{\mu a} u^b \nabla_a \nabla_\mu T_b^\mu = 0$ $P_a^\mu \nabla^2 \nabla_\nu T^{\nu a} = 0$ $P^{\mu a} \nabla_a \nabla_\alpha \nabla_\beta T^{\alpha\beta} = 0$ $P_a^\mu (u.\nabla)^2 \nabla_\nu T^{\nu a} = 0$	$P^{\mu a} (u.\nabla) \nabla_a \Theta$
Pseudo-vectors (1)	$P^{\mu a} (u.\nabla)^2 l_a$ $P^{\mu a} \nabla^2 l_a$	$u_\nu \epsilon^{\mu\nu\alpha\beta} (u.\nabla) \nabla_\alpha \nabla_a T_\beta^a = 0$	$P^{\mu a} \nabla^2 l_a$

The third order curvature data consists of derivatives of the Reimann curvature constrained by Bianchi identity

$$\epsilon^{abcd} \nabla_b R_{\alpha\beta cd} = 0$$

In Table 5 we list the independent curvature data that transforms in the scalar, vector and pseudo vector representations (again we will not need and so do not list the remaining representations)

Finally, fourth order scalar data (all we will need at fourth order), both fluid as well as curvature, is tabuated in Table 6.

Table 5. I_3 type curvature data

	Before imposing eom	Eoms	Independent data
Scalars (2)	$(u.\nabla)R$ $(u.\nabla)R_{00}$ $u_a\nabla_\mu R^{a\mu}$	$u^\mu\epsilon_{\mu a\alpha\beta}\epsilon^{abcd}\nabla_b R^{\alpha\beta}_{cd} = 0$	$(u.\nabla)R_{00}$ $(u.\nabla)R$
Vectors (3)	$P^{\mu a}\nabla_a R_{00}$ $P^{\mu a}\nabla_a R$ $P_a^\mu\nabla_\nu F^{\nu a}$ $P_a^\mu\nabla_\nu R^{\nu a}$ $P^{\mu a}u^b(u.\nabla)R_{ab}$	$u_a u^\nu\epsilon_{\mu\nu\alpha\beta}\epsilon^{abcd}\nabla_b R^{\alpha\beta}_{cd} = 0$ $u_\alpha u^\nu\epsilon_{\mu\nu a\beta}\epsilon^{abcd}\nabla_b R^{\alpha\beta}_{cd} = 0$	$P^{\mu a}\nabla_a R_{00}$ $P_a^\mu\nabla_\nu R^{\nu a}$ $P^{\mu a}u^b(u.\nabla)R_{ab}$
Pseudo-vectors (1)	$u^p u_a \epsilon^{abcd}\nabla_b R^\mu_{pcd}$ $u^p u_a \epsilon^{\mu abc}\nabla_b R_{pc}$	$u_\alpha u_a \epsilon^{abcd}\nabla_b R^{\alpha\mu}_{cd} = 0$	$u^p u_a \epsilon^{\mu abc}\nabla_b R_{pc}$

Table 6. I_4 type scalars

	Before imposing eom	Eoms	Independent data
Fluid data (1)	$(u.\nabla)^3\Theta$ $(u.\nabla)\nabla^2\Theta$ $(u.\nabla)^4T$ $(u.\nabla)^2\nabla^2T$ $\nabla^2(\nabla^2T)$	$(u.\nabla)^3(u_\nu\nabla_\mu T^{\mu\nu}) = 0$ $(u.\nabla)\nabla^2(u_\nu\nabla_\mu T^{\mu\nu}) = 0$ $(u.\nabla)^2\nabla_\mu\nabla_\nu T^{\mu\nu} = 0$ $\nabla^2\nabla_\mu\nabla_\nu T^{\mu\nu} = 0$	$(u.\nabla)^3\Theta$
Curvature data (4)	$(u.\nabla)^2R_{00}$ $(u.\nabla)^2R$ ∇^2R ∇^2R_{00} $u_a(u.\nabla)\nabla_b R^{ab}$ $\nabla_a\nabla_b R^{ab}$ $\nabla_a\nabla_b F^{ab}$	$u_a u^\nu\epsilon_{\mu\nu\alpha\beta}\nabla^\mu\epsilon^{abcd}\nabla_b R^{\alpha\beta}_{cd} = 0$ $u_\alpha u^\nu\epsilon_{\mu\nu a\beta}\nabla^\mu\epsilon^{abcd}\nabla_b R^{\alpha\beta}_{cd} = 0$ $u^\mu\epsilon_{\mu a\alpha\beta}(u.\nabla)\epsilon^{abcd}\nabla_b R^{\alpha\beta}_{cd} = 0$	$(u.\nabla)^2R_{00}$ $(u.\nabla)^2R$ ∇^2R ∇^2R_{00}

3.2 Composite Expressions

In the sequel we will sometimes need to list for example, all 3rd order vectors. In addition to expressions constructed out of the independent 3rd order data, listed in the previous subsection, the set of all 3rd order vectors includes expressions cubic in first order data, and expressions formed out of the product of one first order and one second order piece of data. We will refer to expressions constituted out of products of independent data as composite expressions. Composite expressions

formed out of independent data are easily enumerated and decomposed into $SO(3)$ representations using Clebsh Gordan decompositions (taking care to account for symmetry properties when we multiply two or three copies of the same data).

In order to ease the process of reference to composite expressions in the rest of the note we now adopt the following terminology. Independent data at m^{th} order in the derivative expansion is referred to data of the type I_m . A composite expression that consists of a product of three first order pieces of data is referred to as an expression of the type $C_{1,1,1}$. Composite expressions that consist of the product of a first order and 3rd order piece of data are called expressions of the form $C_{1,3}$. The generalization of our notation to other forms of composite data is obvious.

In the sequel we will need to list only those composite expressions that transform in the vector (in order to list the most general entropy current) or the scalar (in order to list the most general terms in its divergence). In the rest of this subsection we present a listing of those vector and scalar composite expressions that will be needed below.

Table 7. $C_{1,1}$ type expressions

Scalars (4)	$\Theta^2, \mathbf{a}^2, \omega^2, \sigma^2$
Vectors (3)	$\mathbf{a}^\mu \Theta, \mathbf{a}_\nu \omega^{\mu\nu}, \mathbf{a}_\nu \sigma^{\mu\nu}$
Tensors (5)	$\Theta \sigma_{\mu\nu}, \sigma_{\langle\mu}^a \sigma_{a\nu\rangle}, \omega_{\langle\mu}^a \sigma_{a\nu\rangle}, \omega_{\langle\mu}^a \omega_{a\nu\rangle}, \mathbf{a}_{\langle\mu} \mathbf{a}_{\nu\rangle}$

Here $\omega_{\mu\nu} = P^{\mu a} P^{\nu b} \left[\frac{\nabla_a u_b - \nabla_b u_a}{2} \right]$

Table 8. $C_{1,2}$ type expressions independent of the curvature

Scalars (4)	$\Theta(u.\nabla)\Theta, (\nabla_\mu T)\nabla^2 u^\mu, (\nabla_\mu T)(u.\nabla)(\nabla^\mu T), \sigma_{\mu\nu}\nabla^\mu\nabla^\nu T$
Vectors (11)	$P_\mu^\alpha \Theta \nabla^2 u^\mu, P_\mu^\alpha \Theta (u.\nabla) \nabla^\mu T, P_\mu^\alpha (\nabla^2 T) \nabla^\mu T, \omega_{\mu\nu} \nabla^2 u^\mu$ $\omega_{\mu\nu} (u.\nabla) \nabla^\nu T, \sigma_{\mu\nu} \nabla^2 u^\mu, \sigma_{\mu\nu} (u.\nabla) \nabla^\nu T, P_\mu^\alpha (\nabla_a T) (\nabla^\mu \nabla^a T)$ $(\nabla_a \omega_{b\mu}) \sigma^{ab}, P_\mu^\alpha \sigma_{ab} \nabla^a \nabla^b u^\mu, P_\mu^\alpha \omega^{ab} \nabla^\mu \omega_{ab}$

Table 9. $C_{1,2}$ type expressions involving a curvature

Scalars (5)	$F_{ab} \sigma^{ab}, R_{ab} \sigma^{ab}, u_a \mathbf{a}_b R^{ab}, \Theta R, \Theta R_{00}$
Vectors (9)	$P_\mu^\alpha \mathbf{a}_\nu F^{\mu\nu}, P_\mu^\alpha \mathbf{a}_\nu R^{\mu\nu}, \mathbf{a}^\mu R_{00}, \mathbf{a}^\mu R, P_\mu^\alpha u_a \Theta R^{a\mu}$ $u^a R_{ab} \sigma^{b\mu}, u_a R^{a\mu bc} \omega_{bc}, P_\mu^\alpha u_a R^{ab\mu c} \sigma_{bc}, u^a R_{ab} \omega^{b\mu}$

Table 10. $C_{1,1,1}$ type expressions

Scalars (7)	$\Theta^3, \sigma^2\Theta, \omega^2\Theta, \mathfrak{a}^2\Theta$ $\mathfrak{a}_\mu\mathfrak{a}_\mu\sigma^{\mu\nu}, \sigma_{\mu a}\sigma_b^a\sigma^{b\mu}, \omega_{\mu a}\sigma_b^a\omega^{b\mu}$
Vectors (10)	$\sigma^2\mathfrak{a}^\mu, \omega^2\mathfrak{a}^\mu, \Theta^2\mathfrak{a}^\mu, \mathfrak{a}^2\mathfrak{a}^\mu, \Theta\sigma^{\mu\nu}\mathfrak{a}_\nu$ $\Theta\omega^{\mu\nu}\mathfrak{a}_\nu, \sigma^{\mu a}\sigma_{ab}\mathfrak{a}^b, \omega^{\mu a}\omega_{ab}\mathfrak{a}^b, \omega^{\mu a}\sigma_{ab}\mathfrak{a}^b, \sigma^{\mu a}\omega_{ab}\mathfrak{a}^b$

4 Entropy current

In this section we will derive constraints on constitutive relations, at second order in the derivative expansion, from the requirement of positivity of divergence of *any* entropy current that reduces to su^μ (where s is the entropy density) in equilibrium.

We will first explain in very broad terms how we proceed.

The entropy current takes the form

$$J^\mu = J_{eq}^\mu + \tilde{J}^\mu \quad (4.1)$$

where

$$J_{eq}^\mu = su^\mu$$

In general \tilde{J}^μ has terms of all orders in the derivative expansion, but for the purposes of this note we will find it sufficient to truncate \tilde{J}^μ to terms of third order or lower in derivatives. To start with we allow \tilde{J} to be given by the most general possible form consistent with symmetries. We then compute the divergence of J^μ and reexpress the final result entirely in terms of independent data of fourth or lower order in derivatives. The last step (reexpressing the divergence of J in terms of independent data) uses the equations of fluid dynamics, and so the constitutive relations. Our final expression is a polynomial in the (finite number of) pieces of data of fourth or lower order in derivatives. We then demand that the resultant polynomial is positive definite (or can be made so by the addition of terms higher than fourth order in the derivative expansion) as a function of its arguments. This rather stringent requirement turns out to yield several constraints on the form of both the entropy current at second (and third) order as well as constitutive relations at second order in the derivative expansion.

4.1 Entropy current in equilibrium and 1st order correction

In order to set the stage for our discussion we first recall how the requirement of positivity of the entropy current constrains constitutive relations at first order in the derivative expansion [7]. We first recall that thermodynamics and the fluid dynamical equations may be used to demonstrate that

$$\nabla_\mu J_{eq}^\mu = -\frac{1}{T} \left(\Pi^{\mu\nu} \sigma_{\mu\nu} + \frac{\Theta \Pi_\mu^\mu}{3} \right) \quad (4.2)$$

It follows in particular from (4.2) that entropy is conserved in perfect fluid dynamics (i.e. when $\Pi^{\mu\nu}$ vanishes). It also follows that the divergence of the most general entropy current (4.1) only contains terms of second or higher order in the derivative expansion. Let us now examine the constraints from the requirement of positivity of these second order pieces. For this purpose we need to study the most general entropy current at first order in derivatives. Imposing the requirement of invariance under parity, the most general (onshell inequivalent) family of first order entropy currents is given by

$$J^\mu = J_{\text{equilibrium}}^\mu + \alpha \Theta u^\mu + \beta \mathbf{a}^\mu \quad (4.3)$$

(we have used here that at first order in the derivative expansion we have one piece of scalar data, which may be chosen as Θ , and one piece of vector data, which may be chosen as \mathbf{a}^μ). We now proceed to compute the divergence of (4.3) and use the perfect fluid equations to reexpress the result in terms of independent data. The resultant expression is the sum of a quadratic form in first order data and a linear form in 2nd order scalar data. As derived in [7], the final expression for this divergence is given as

$$\begin{aligned} & \nabla_\mu J^\mu|_{\text{upto 2nd order}} \\ &= -\frac{1}{T} \left(\Pi^{\mu\nu} \sigma_{\mu\nu} + \frac{\Pi \Theta}{3} \right) \\ & \quad + \Theta (u \cdot \nabla) \alpha + (\mathbf{a} \cdot \nabla) \beta + \left(\alpha + \frac{\beta}{3} \right) \frac{\Theta^2}{3} + \beta (\sigma^2 + \omega^2) \\ & \quad + (\alpha + \beta) (u \cdot \nabla) \Theta + \beta R_{00} \end{aligned} \quad (4.4)$$

Here both the first and the second line have terms quadratic in 1st order data. The last line contains the terms which are linear in second order data. There are three independent 2nd order scalars ($(u \cdot \nabla) \Theta$, R , R_{00}) as given in the classification in section 3. Only two of these three scalars appear in (4.4). Since these two terms are linear in fluid variable, they can have any sign and to ensure positivity of the divergence their coefficients (both α and β) must be set to zero. This implies that at 1st order no correction can be added to the entropy current which is consistent with the positivity requirement. Then in the RHS of (4.4) only the first line will give a non-zero contribution. To evaluate the first line we need the first order corrections to the constitutive relation. At first order the most general correction to the constitutive relation (stress tensor in Landau gauge) will involve the single on-shell independent 1st order scalar which we have chosen to be Θ and single on-shell independent tensor $\sigma_{\mu\nu}$.

$$\Pi_{\mu\nu}|_{\text{upto 1st order}} = -\eta \sigma_{\mu\nu} - \zeta \Theta P_{\mu\nu}$$

where η and ζ are shear and bulk viscosity respectively.

Therefore finally

$$\nabla_\mu J^\mu|_{\text{upto 2nd order}} = \frac{1}{T} (\eta \sigma^2 + \zeta \Theta^2) \quad (4.5)$$

Hence to have a positive definite divergence one requires that

$$\eta \geq 0, \quad \zeta \geq 0$$

The main point to note in the above equation (4.5) is that it involves only two of the four first order on-shell independent data as listed in section 3. The squares of the independent vector \mathfrak{a}^μ and the pseudo-vector l^μ do not appear in equation (4.5). Because of this fact any term in the divergence which is of the form $(\mathfrak{a}_\mu \times I_2 \text{ or } I_3 \text{ type vector})$ or $(l_\mu \times I_2 \text{ or } I_3 \text{ type pseudo-vector})$ can never be made positive-definite.

4.2 General constraints on second and the third order corrections

In general \tilde{J}^μ can be written as

$$\tilde{J}^\mu = \left(\sum_i \mathfrak{S}_i \right) u^\mu + \sum_i \mathfrak{V}_i^\mu$$

where \mathfrak{S}_i is an arbitrary combination of i th order on-shell independent scalars and \mathfrak{V}_i^μ is a combination of i th order vectors. In the previous subsection we have seen that to constrain the first order transport coefficients η and ζ we need to determine only the first order correction to the entropy current (i.e. only \mathfrak{S}_1 and \mathfrak{V}_1^μ and both of them finally turn out to be zero). But to constrain the second order transport coefficients we need to go till the third corrections to the entropy current. The reason is the following.

Suppose the divergence of the most general entropy current has two terms of the form

$$\nabla_\mu J_s^\mu = Ax^2 + Bxy = Ax^2 \left(1 + \frac{By}{Ax} \right)$$

where x and y are two on-shell independent fluid data and A and B are some functions of temperature, which in general will depend on the coefficients appearing in the entropy current or transport coefficients.

In this schematic expression of divergence since x and y are two independent fluid data, locally the ratio $\frac{By}{Ax}$ can take any negative value, larger or smaller than 1 in magnitude and the positivity constraint will depend on whether y^2 term is present or not in the final expression of the divergence. In the absence of a y^2 piece, the coefficient B has to be set to zero and the coefficient A to some non-negative number.

But this argument does not require x and y to be of same order in derivative expansion. Even when x is of first order in derivative and y is of second order,

the ratio $\frac{By}{Ax}$ can be of order 1 for some particular fluid configuration where x is accidentally small enough to be comparable to y at a given point. In such cases to see whether y^2 term is present or not, we need to compute the divergence till fourth order. This is why we have to compute the divergence till fourth order even if we want to constrain just the second order transport coefficients.

In fact the constraints on transport coefficients will involve situation where x and y are necessarily of different orders. For example, B will contain some second order transport coefficients only when x is of first order (as we will see below that x has to be equal to $\sigma_{\mu\nu}$ or Θ) and y is of second order in derivatives. It will turn out that most of the equalities among the coefficients will follow from this sort of argument.

Below we schematically list all the constraints we need to impose on the third and fourth order pieces of the divergence in order to ensure its positivity.

- The coefficient of any term (appearing in third or fourth order piece of the divergence) which contains more than one factors of $\sigma_{\mu\nu}$ or Θ or at least one factor of $\Theta\sigma_{\mu\nu}$ will not have any constraint from positivity as long as η and ζ are non-zero and are of order one. This is because whenever such third or fourth order terms are non-zero, the second order piece of the divergence is also non-zero and positive-definite and will always dominate these terms within derivative expansion. These terms can never make the divergence negative. Therefore while calculating the the third and fourth order divergence we shall ignore all these terms.
- One needs to do sixth order analysis to constrain the coefficients of any term (appearing in the fourth order divergence of the entropy current) which is of the form $(\sigma_{\mu\nu} \times \text{Some third order tensor})$ or $(\Theta \times \text{Some third order scalar})$. Such terms generically will have contributions from third order transport coefficients. Since we are interested only upto second order transport coefficients we shall ignore all such terms while calculating the fourth order divergence.
- The coefficients of all the terms which contain a single I_2 , I_3 or I_4 type scalar (at second, third and fourth order respectively) have to be set to zero. This is because locally all these terms are linear in fluid variables and therefore can have any sign.
- In the third order and fourth order piece of the divergence, the coefficients of all the terms which are of the form $(\mathbf{a}_\mu \times I_2 \text{ or } I_3 \text{ type vector})$ or $(l_\mu \times I_3 \text{ type pseudo-vector})$ have to be set to zero. Since there is no on-shell independent I_2 type pseudo-vector, there could not be any term of the form $(l_\mu \times I_2 \text{ type pseudo-vector})$.

- At this stage the terms appearing in the third order piece of the divergence will be of the following form.

1. $\sigma_{\mu\nu} \times (I_2 \text{ or } C_{1,1} \text{ type tensors})$
2. $\Theta \times (I_2 \text{ or } C_{1,1} \text{ type scalars})$

All these terms will involve the second order transport coefficients.

The relevant terms appearing at the fourth order (where all the terms involving $\sigma_{\mu\nu}$ and Θ are ignored) will be of the following form.

1. A quadratic form involving independent I_2 type data
2. A quartic form involving \mathfrak{a}_μ and $\omega_{\mu\nu}$
3. Terms linear in I_2 type data and quadratic in \mathfrak{a}_μ and/or $\omega_{\mu\nu}$

Therefore when $\eta \neq 0$ and $\zeta \neq 0$ the relevant part of the divergence calculated upto fourth order is schematically given by

$$\begin{aligned}
\text{Divergence} = & \frac{\eta \sigma^2 + \zeta \Theta^2}{T} \\
& + \sigma_{\mu\nu} \times (I_2 \text{ or } C_{1,1} \text{ type tensors}) + \Theta \times (I_2 \text{ or } C_{1,1} \text{ type scalars}) \\
& + \text{A quadratic form involving independent } I_2 \text{ type data} \\
& + \text{Terms linear in } I_2 \text{ type data and quadratic in } \mathfrak{a}_\mu \text{ and/or } \omega_{\mu\nu} \\
& + \text{A quartic form involving } \mathfrak{a}_\mu \text{ and } \omega_{\mu\nu}
\end{aligned} \tag{4.6}$$

where in the second line all the $C_{1,1}$ type tensors involving $\sigma_{\mu\nu}$ and all the $C_{1,1}$ type scalars involving Θ are ignored.

- Now we can shift $\sigma_{\mu\nu}$ by a combination of I_2 or $C_{1,1}$ type tensors such that the term linear in $\sigma_{\mu\nu}$ appearing in the second line gets absorbed. This shift will generate fourth order terms structurally similar to the terms appearing in third, fourth and fifth line of the above equation. One can see that all these newly generated terms together will necessarily be negative definite. Similar shift has to be done to absorb the terms linear in Θ to the first line of (4.6).
- One can do similar shifts in I_2 type data to absorb the terms appearing in the fourth line of equation (4.6) into terms appearing in the third and fifth line with I_2 data replaced by the shifted one. At this stage the schematic expression of

the divergence will take the following form.

$$\begin{aligned} \text{Divergence} = & \frac{\eta (\text{shifted } \sigma)^2 + \zeta (\text{shifted } \Theta)^2}{T} \\ & + \text{A quadratic form involving shifted } I_2 \text{ type data} \\ & + \text{A quartic form involving } \mathbf{a}_\mu \text{ and } \omega_{\mu\nu} \end{aligned} \quad (4.7)$$

- The positive definiteness of the divergence finally will imply the positivity of the quadratic and the quartic form appearing in the second and the third line of (4.7).

Such condition will generically give some inequalities among the coefficients. However suppose by explicit computation one finds that for some particular negative definite term generated by the shift there is no term present in the third or fifth line of equation (4.6) to compensate. Then this will imply that the coefficient of the corresponding linear term (the source for generating this particular negative-definite term through shift) in the second line or fourth line has to be set to zero. This will give strict equalities among the coefficients.

It will turn out that all of the constraints on the 2nd order transport coefficients will arise from this sort of argument.

In explicit calculation we will see that in the quadratic form involving the I_2 type data there will not be any term proportional to R_{00}^2 , R^2 , $F_{\mu\nu}F^{\mu\nu}$ and $R_{\mu\nu}R_{ab}P^{\mu a}P^{\nu b}$. This will imply that the coefficients of all the terms linear in R_{00} , R , $F_{\mu\nu}$ and $R_{\mu\nu}P^{\mu a}P^{\nu b}$ have to be zero. It turns out that once we set these linear terms to zero, the quartic form mentioned in the last line of equation (4.6) also vanishes.

The vanishing of these terms at fourth order gives the final constraint on the transport coefficients. In the explicit computation we will see that there are eight terms ($\Theta\mathbf{a}^2$, Θl^2 , $\sigma_{\mu\nu}\mathbf{a}^\mu\mathbf{a}^\nu$, $\sigma_{\mu\nu}l^\mu l^\nu$, $\sigma_{\mu\nu}R^{\mu\nu}$, $\sigma_{\mu\nu}F^{\mu\nu}$, $R\Theta$ and $R_{00}\Theta$) in the third order divergence which are linear in the set of fluid and curvature data mentioned above and also involve eight independent transport coefficients. So setting the coefficient of these linear terms to zero, we can express the eight transport coefficients in terms of the coefficients appearing in the second order entropy current. It will turn out only three of the entropy current coefficients appear in these expressions. Eliminating these three coefficients we get the final five relations among the 15 transport coefficients as presented in (1.3).

Once all these relations are imposed on the divergence, one is left with a quadratic form involving only I_2 type data. To ensure that this quadratic form is positive-definite the coefficients appearing in the second and third order entropy current as well as the transport coefficients have to satisfy some inequalities. But in this case, at least upto this order the entropy current coefficients can not be eliminated from the relations. Therefore unlike the first order transport coefficients the second order ones do not satisfy any inequalities within themselves.

4.3 Implementing the general rules at second order

At second order we have to determine \mathfrak{S}_2 and \mathfrak{V}_2^μ such that the divergence calculated upto third order in derivative expansion is non-negative. Here we shall follow the general procedure described in the previous subsection.

We shall express \mathfrak{S}_2 and \mathfrak{V}_2^μ in terms of the on-shell independent second order scalars and vectors respectively. \mathfrak{S}_2 will have 7 coefficients, three multiplying the three independent I_2 type scalars and rest of four multiplying the four $C_{1,1}$ type scalars. \mathfrak{V}^μ will also have 6 coefficients, three multiplying the three I_2 type vectors and the rest multiplying the three $C_{1,1}$ type vectors.

So before imposing any constraint the entropy current at second order contains total 13 coefficients, each of which is an arbitrary function of temperature.

We shall write this most general 13 parameter entropy current in the following form .

$$\begin{aligned} \tilde{J}^\mu|_{\text{second order}} = & \nabla_\nu [A_1(u^\mu \nabla^\nu T - u^\nu \nabla^\mu T)] + \nabla_\nu (A_2 T \omega^{\mu\nu}) \\ & + A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu + [A_4 (u \cdot \nabla) \Theta + A_5 R + A_6 R_{00}] u^\mu \\ & + (B_1 \omega^2 + B_2 \Theta^2 + B_3 \sigma^2) u^\mu + B_4 [(\nabla s)^2 u^\mu + 2s \Theta \nabla^\mu s] \\ & + [\Theta \nabla^\mu B_5 - P^{ab} (\nabla_b u^\mu) (\nabla_a B_5)] + B_6 \Theta \mathfrak{a}^\mu + B_7 \mathfrak{a}_\nu \sigma^{\mu\nu} \end{aligned} \quad (4.8)$$

Here s is the entropy density and all the coefficients A_i and the B_i are the arbitrary functions of temperature.

Now we shall argue that (4.8) is actually the most general 13 parameter entropy current. By equations of motion one can show that the only I_2 type vector appearing in the first term is $P^{\mu a} \nabla_a \Theta$, the second term contains a linear combination of all the three independent I_2 type vectors and in the third term the only I_2 type vector that appears is $P^{\mu a} R_{ab} u^b$. Therefore the first three terms together take care of the all the three I_2 type vectors. Terms multiplying A_4 , A_5 and A_6 are the three I_2 type scalars.

By equation of motion B_4 term is equal to a linear combination of $\Theta^2 u^\mu$, $\mathfrak{a}^2 u^\mu$ and $\Theta \mathfrak{a}^\mu$. Similarly B_5 term is a particular linear combination of $\Theta^2 u^\mu$, $\mathfrak{a}^2 u^\mu$, $\Theta \mathfrak{a}^\mu$, $\mathfrak{a}_\nu \sigma^{\mu\nu}$ and $\mathfrak{a}_\nu \omega^{\mu\nu}$. Therefore all the $C_{1,1}$ type scalars and vectors appear in (4.8) with distinct coefficients.

Next we shall compute the divergence of this 13 parameter entropy current constructed in (4.8). We have to set the coefficients of all the I_3 type on-shell independent terms to zero. Since there are total 3 independent I_3 type scalars, it can impose at most three relations among the coefficients appearing in the second order entropy current. Next we have to isolate all the $C_{1,2}$ type terms which are of the form of \mathfrak{a}_μ times a I_2 type vector and set their coefficients to zero. Since there are total three second order I_2 type vectors this condition also can impose at most three constraints.

- The divergence of the first two terms in (4.8) vanish identically.

- The divergence of the third term (the term with coefficient A_3) does not produce any I_3 type scalar. The divergence of this term is explicitly computed in (B.2).
- The three independent I_3 type scalars ($u^a u^b \nabla_a \nabla_b \Theta$, $u \cdot \nabla R$, $u \cdot \nabla R_{00}$) are produced from the three terms multiplying coefficients A_4 , A_5 and A_6 respectively. Therefore to maintain positivity A_4 , A_5 and A_6 have to be set to zero.
- The divergence of the terms multiplying B_1 , B_2 , B_3 and B_4 do not produce any term of the form \mathfrak{a}^μ times an I_2 type vector. The divergence of these terms are explicitly computed in (B.5), (B.6), and (B.7) respectively.
- The divergence of the terms with coefficients B_6 and B_7 produce the two terms $\mathfrak{a}_\nu \nabla_\mu \sigma^{\mu\nu}$ and $\mathfrak{a}_\mu \nabla^\mu \Theta$ respectively whose net coefficient should be zero to ensure the positivity of the divergence.

Since these are the only places where these terms are produced, B_6 and B_7 are set to zero.

- Both the terms multiplying B_5 and A_3 produce the third possible term of the form \mathfrak{a}^μ times an I_2 type vector which is $\mathfrak{a}_\mu R^{\mu\nu} u_\nu$ (see (B.2) and (B.3)). The net coefficient is $\left(\frac{A_3}{T} + \frac{dB_5}{dT} - \frac{dB_5}{dT}\right)$.

Therefore positivity implies

$$\frac{dB_5}{dT} = \frac{A_3}{T} + \frac{dA_3}{dT}$$

After imposing all these constraints the final form of the second order entropy current is given as

$$\begin{aligned} & \tilde{J}^\mu|_{\text{second order}} \\ &= \nabla_\nu [A_1(u^\mu \nabla^\nu T - u^\nu \nabla^\mu T)] + \nabla_\nu (A_2 T \omega^{\mu\nu}) \\ &+ A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu + \left(\frac{A_3}{T} + \frac{dA_3}{dT} \right) [\Theta \nabla^\mu T - P^{ab} (\nabla_b u^\mu) (\nabla_a T)] \\ &+ (B_1 \omega^2 + B_2 \Theta^2 + B_3 \sigma^2) u^\mu + B_4 [(\nabla s)^2 u^\mu + 2s \Theta \nabla^\mu s] \end{aligned} \quad (4.9)$$

4.4 Implementing the general rules at third order

- First we have to write \mathfrak{S}_3 and \mathfrak{V}_3^μ in terms of the on-shell independent data.
- The coefficients of all the I_4 type data appearing at the fourth order divergence have to be set to zero. An I_4 type term in the fourth order divergence can occur only when the derivative acts on the I_3 type terms of the third order entropy current. Therefore this condition will constrain the coefficients of the I_3 type terms.

Now there are 3 I_3 type independent third order scalars and 4 I_3 type independent third order vectors and there are total 5 I_4 type fourth order scalars. This means one can have at least 2 distinct coefficients multiplying I_3 type terms in a third order entropy current with positive definite divergence.

It turns out that after imposing this constraint there are exactly two coefficients left and the terms multiplying them can be chosen in such a way that their divergence vanish identically. So these two terms will not contribute to any further constraint.

- The number of free coefficients that can multiply the $C_{1,2}$ type data in third order entropy current is quite large. There can be total 9 free coefficients in \mathfrak{S}_3 and 20 in \mathfrak{V}_3^μ .

These coefficients will be constrained by the fact that any terms of the form \mathfrak{a}_μ times a I_3 type vector or l_μ times a I_3 type pseudo-vector have to be set to zero.

Since there are total 4 I_3 type vectors and total 2 I_3 pseudo-vectors, it can produce at most 6 constraints, reducing the number of free coefficients to 23.

- Since there is no other general constraint to simplify the form of the entropy current at this stage we have to calculate the divergence.

But we will not attempt to calculate the full divergence. Instead we shall calculate only those terms which can impact the constraints on the second order transport coefficients.

For this purpose in the divergence we can ignore all those terms which are multiplied by Θ or $\sigma_{\mu\nu}$.

Also to simplify we shall try to write the terms in the form of $(\nabla_\mu A^{\mu\nu})$ where $A^{\mu\nu}$ is an anti-symmetric tensor, so that their divergence vanish identically. We could not do it for all the independent terms, but we try to apply this trick for as many terms as possible.

Now we shall explicitly construct the required part of the third order entropy current piece by piece.

The part multiplying the I_3 type terms can be written as

$$\begin{aligned}
& \tilde{J}^\mu|_{\text{3rd order}/I_3\text{type}} \\
&= \nabla_\nu [P_1(u^\nu(u.\nabla)\nabla^\mu T - u^\mu(u.\nabla)\nabla^\nu T)] + \nabla_\nu [P_2(u^\nu R_\theta^\mu u^\theta - u^\mu R_\theta^\nu u^\theta)] \\
& \quad + [P_3(u.\nabla)^3 T + P_4(u.\nabla)R + P_5(u.\nabla)R_{00}] u^\mu \\
& \quad + P^{\mu a} [P_6 \nabla_a R_{00} + P_7 \nabla_a R]
\end{aligned} \tag{4.10}$$

In the first and the third term the two independent I_3 type fluid data are chosen to be $(u.\nabla)^2 \nabla^\mu T$ and $(u.\nabla)^3 T$. The independent I_3 type curvature data are chosen

from the list given in section 3. The second term (with the coefficient P_2) contains the independent vector $P^{\mu a} u^b (u \cdot \nabla) R_{ab}$.

Here all the terms in the third and the fourth line produce independent I_4 type scalars at fourth order and therefore they are all set to zero. The divergence of the first two terms vanish identically.

So finally this part of the entropy current has only two terms and both the terms have zero divergence.

$$\begin{aligned} & \tilde{J}^\mu \big|_{\text{3rd order}/I_3 \text{ type}} \\ &= \nabla_\nu [P_1(u^\nu (u \cdot \nabla) \nabla^\mu T - u^\mu (u \cdot \nabla) \nabla^\nu T)] + \nabla_\nu [P_2(u^\nu R_\theta^\mu u^\theta - u^\mu R_\theta^\nu u^\theta)] \end{aligned} \quad (4.11)$$

The part multiplying the $C_{1,2}$ type terms have total 29 coefficients to begin with. We shall try to write them in a way so that the computation becomes simpler. In table (11) and table (12) we have listed each of the independent $C_{1,2}$ type fluid data and then the independent combination through which this data has entered the entropy current. In table (15) we have listed the relevant $C_{1,1,1}$ type scalars and vectors and also their coefficients in the entropy current. In all these cases, to begin with the coefficients are some unspecified functions of temperature.

Table 11. $C_{1,2}$ type Scalars (Fluid data)

Scalars as listed before	Combination that enters entropy current
$\Theta(u \cdot \nabla) \Theta$	$Q_1 [\Theta(u \cdot \nabla) \Theta] u^\mu$
$\sigma_{\mu\nu} (u \cdot \nabla) \sigma^{\mu\nu}$	$Q_2 [\sigma_{ab} (u \cdot \nabla) \sigma^{ab}] u^\mu$
$\mathbf{a}_\mu \nabla_a \sigma^{a\mu}$	$\nabla_\mu [Q_3 (u^\mu \sigma^{a\nu} - u^\nu \sigma^{a\mu}) \mathbf{a}_a]$
$\mathbf{a}_\mu \nabla_a \omega^{a\mu}$	$\nabla_\mu [Q_4 (u^\mu \omega^{a\nu} - u^\nu \omega^{a\mu}) \mathbf{a}_a]$

We shall start our analysis by computing the divergence of the terms appearing in table (11) and (12).

- The divergence of the terms with coefficients (Q_i , $i = 3, \dots, 7$) vanish identically.
- It turns out that the term with coefficient Q_8 is the only term which produces \mathbf{a}_μ times a third order I_3 type vector. Therefore Q_8 has to be set to zero.
- Similarly if we analyse only the third order entropy current, the term with coefficient Q_9 is the only term that produces l_μ times a third order I_3 type pseudo-vector in the fourth order divergence. However a similar term is also produced when the divergence of the B_1 term in the second order entropy current is computed upto fourth order.

$$\nabla_\mu [B_1 \omega^2 u^\mu] = B_1 \omega^2 \Theta + [(u \cdot \nabla) B_1] \omega^2 + 2B_1 \omega^{ab} (u \cdot \nabla) \omega_{ba} \quad (4.12)$$

Table 12. $C_{1,2}$ type Vectors (Fluid data)

Vectors as listed before	Combination that enters entropy current
$\Theta \nabla^\mu \Theta$	$\nabla_\nu [Q_5 \Theta (u^\mu g^{a\nu} - u^\nu g^{a\mu}) \mathbf{a}_a]$
$\Theta \nabla_\nu \omega^{\mu\nu}$	$\nabla_\nu [Q_6 \Theta \omega^{\mu\nu}]$
$\omega^{\mu\nu} \nabla_a \sigma_\nu^a$	$\nabla_\nu [Q_7 (\omega^{\mu\theta} \sigma_\theta^\nu - \omega^{\nu\theta} \sigma_\theta^\mu)]$
$\mathbf{a}_a (\nabla^\mu \nabla^a T)$	$Q_8 \mathbf{a}_a (\nabla^\mu \nabla^a T)$
$\omega^{ab} \nabla^\mu \omega_{ab}$	$Q_9 \omega^{ab} \nabla^\mu \omega_{ab}$
$\mathbf{a}^\mu (u \cdot \nabla) \Theta$	$Q_{10} [\mathbf{a}^\mu (u \cdot \nabla) \Theta - u^\mu (\mathbf{a} \cdot \nabla) \Theta]$
$\omega_{\mu\nu} \nabla^\nu \Theta$	$\omega^{\mu\nu} \nabla_\nu (Q_{11} \Theta)$
$\sigma_\nu^\mu \nabla^\nu \Theta$	$Q_{12} \sigma_\nu^\mu \nabla^\nu \Theta$
$\sigma_\nu^\mu \nabla_\theta \sigma^{\theta\nu}$	$Q_{13} \sigma_\nu^\mu \nabla_\theta \sigma^{\theta\nu}$
$P_c^\mu \sigma_{ab} \nabla^{<c} \sigma^{ab>}$	$P_c^\mu \sigma_{ab} \nabla^{<c} \sigma^{ab>}$
$P^{\mu c} \sigma^{ab} (\nabla_a \omega_{bc} - \frac{P_{ab}}{3} \nabla^k \omega_{kc})$	$Q_{15} P^{\mu c} \sigma^{ab} (\nabla_a \omega_{bc} - \frac{P_{ab}}{3} \nabla^k \omega_{kc})$

Since there is no second order on-shell pseudo vector, $\omega^{ba}(u \cdot \nabla) \omega_{ba}$ must contain a third order pseudo-vector times l_μ . Then in the final fourth order divergence the total coefficient of such term (ie. the term proportional to third order pseudo-vector times l_μ) will be a linear combination of B_1 and Q_9 , which should be set to zero.

But to simplify the calculation instead we shall introduce a third order shift in the ‘ B_1 term’ of the second order entropy current and will consider the following term $[B_1 \omega^2 u^\mu - \frac{2B_1}{T_s} \omega^{\mu b} \nabla_a \Pi_b^a]$. The divergence of the shifted ‘ B_1 term’ no longer contains the terms proportional to third order pseudo-vector times l_μ . (The relevant part for the divergence of the shifted ‘ B_1 ’ term is computed in (B.5)). Once this shift is done, Q_9 also has to be set to zero, since now this is the only term which produces l_μ times a third order I_3 type pseudo-vector.

For the rest of the 8 terms we have to compute the divergence explicitly. However we are interested in those terms in the fourth order divergence which does not have any explicit factor of Θ or $\sigma_{\mu\nu}$. This simplifies the calculation. For example, in the divergence of the term with coefficient Q_1 , the only contribution which will be relevant for our purpose is $\left(Q_1 [(u \cdot \nabla) \Theta]^2 \right)$.

- The relevant part of the divergence of the last two terms with coefficients Q_{14} and Q_{15} are the following.

$$\begin{aligned}
\nabla_\mu [Q_{14} P_c^\mu \sigma_{ab} \nabla^{(c} \sigma^{ab)}] &\Rightarrow Q_{14} P_\nu^\mu [\nabla_{\langle \mu} \sigma_{ab \rangle}] [\nabla^{\langle \nu} \sigma^{ab \rangle}] \\
\nabla_\mu \left[Q_{15} P^{\mu c} \sigma^{ab} \left(\nabla_a \omega_{bc} - \frac{P_{ab}}{3} \nabla^k \omega_{kc} \right) \right] &\Rightarrow Q_{15} P^{\nu c} [\nabla_\nu \sigma_b^a] \left[\nabla_a \omega_{bc} - \frac{P_{ab}}{3} \nabla^k \omega_{kc} \right]
\end{aligned} \tag{4.13}$$

Here in the first line we get a term proportional to $(\text{spin-3})^2$ and in the second line we get a term proportional to $(\text{pseudo-tensor})^2$. It will turn out that such terms cannot occur in any other place. Positivity of the divergence will be satisfied if both Q_{14} and Q_{15} are positive. Therefore these terms will not produce any constraint on the second order transport coefficients.

- The other five terms where the relevant parts are easy to calculate are the following.

$$\begin{aligned}
\nabla_\mu [Q_1 u^\mu \Theta (u \cdot \nabla) \Theta] &\Rightarrow Q_1 [(u \cdot \nabla) \Theta]^2 \\
\nabla_\mu [Q_2 u^\mu \sigma_{ab} (u \cdot \nabla) \sigma^{ab}] &\Rightarrow Q_2 [(u \cdot \nabla) \sigma_{ab}] [(u \cdot \nabla) \sigma^{ab}]
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
\nabla_\mu [Q_{12} \sigma_\nu^\mu \nabla^\nu \Theta] &\Rightarrow Q_{12} [\nabla_\mu \sigma_\nu^\mu] [\nabla^\nu \Theta] \\
\nabla_\mu [Q_{13} \sigma_\nu^\mu \nabla_\theta \sigma^{\theta \nu}] &\Rightarrow Q_{13} [\nabla_\mu \sigma_\nu^\mu] [\nabla_\theta \sigma^{\theta \nu}]
\end{aligned}$$

- The relevant part in the divergence of the terms with coefficients Q_{10} and Q_{11} are more complicated.

The divergence of the ‘ Q_{11} -term’ is given by the following expression.

$$\begin{aligned}
&\nabla_\mu [\omega^{\mu\nu} \nabla_\nu (Q_{11} \Theta)] \\
&= Q_{11} [\nabla_\mu \omega^{\mu\nu}] [\nabla_\nu \Theta] + [\nabla_\mu \omega^{\mu\nu}] [\nabla_\nu Q_{11}] \Theta \\
&\Rightarrow -Q_{11} [\nabla^\mu \Theta] \left[-P_{\nu\mu} \nabla_a \sigma^{a\nu} + \frac{2}{3} P_{\nu\mu} \nabla^\nu \Theta + P_{\mu\nu} u_a R^{a\nu} + \mathfrak{a}_b \omega^{b\mu} \right] \\
&\quad - Q_{11} \omega^2 (u \cdot \nabla) \Theta
\end{aligned} \tag{4.15}$$

where in the last line we have used the identity (A.5) and ignored the terms proportional to Θ and $\sigma_{\mu\nu}$.

The divergence of the ‘ Q_{10} -term’ is given by

$$\begin{aligned}
& \nabla_\mu \left(Q_{10} [\mathfrak{a}^\mu (u \cdot \nabla) \Theta - u^\mu (\mathfrak{a} \cdot \nabla) \Theta] \right) \\
&= -T \left(\frac{dQ_{10}}{dT} \right) \mathfrak{a}^2 (u \cdot \nabla) \Theta + s \left(\frac{dQ_{10}}{ds} \right) \Theta (\mathfrak{a} \cdot \nabla) \Theta \\
&\quad + Q_{10} [(\nabla \cdot \mathfrak{a})(u \cdot \nabla) \Theta + \mathfrak{a}^\mu (\nabla_\mu u^a) (\nabla_a \Theta) - (\nabla_b \Theta) (u \cdot \nabla) \mathfrak{a}^b - \Theta (\mathfrak{a} \cdot \nabla) \Theta] \\
&\Rightarrow - \left(T \frac{dQ_{10}}{dT} + Q_{10} \right) \mathfrak{a}^2 (u \cdot \nabla) \Theta \\
&\quad + Q_{10} \left[\omega^2 (u \cdot \nabla) \Theta + [(u \cdot \nabla) \Theta]^2 + R_{00} (u \cdot \nabla) \Theta - \left(\frac{s}{T} \frac{dT}{ds} \right) P_{ab} (\nabla^a \Theta) (\nabla^b \Theta) \right]
\end{aligned} \tag{4.16}$$

To express the divergence in the chosen basis of independent data we have used the identities (A.1), (A.3) and (A.4).

Here also in the final expression we have ignored the terms proportional to Θ and $\sigma_{\mu\nu}$.

Table 13. $C_{1,2}$ type Scalars (Curvature data)

Scalars as listed before	Combination that enters entropy current
$F_{ab} \sigma^{ab}$	$P_1 (F_{ab} \sigma^{ab}) u^\mu$
$R_{ab} \sigma^{ab}$	$P_2 (R_{ab} \sigma^{ab}) u^\mu$
ΘR	$P_3 u^\mu \Theta R$
ΘR_{00}	$P_4 u^\mu \Theta R_{00}$
$u_a \mathfrak{a}_b R^{ab}$	$P_5 u^\mu (u_a \mathfrak{a}_b R^{ab})$

Next we shall compute the divergence of the curvature type data appearing in table (13) and (14).

- It turns out that the divergence of the terms multiplying P_5 , P_6 and P_7 are the only terms which produce the terms of the form \mathfrak{a}_μ times an independent third order I_3 type curvature vector. Therefore we have to set P_5 , P_6 and P_7 to zero.
- Similarly the divergence of the term multiplying P_8 is the only place where l_μ times a third order I_3 type curvature pseudo-vector is produced. Therefore P_8 should also be set to zero.

The divergence of the remaining 10 terms have to be computed. Here also we shall ignore any term that are multiplied by an explicit factor of Θ or $\sigma_{\mu\nu}$.

Table 14. $C_{1,2}$ type Vectors (Curvature data)

Vectors as listed before	Combination that enters entropy current
$\mathfrak{a}^\mu R$	$P_6 \mathfrak{a}^\mu R$
$\mathfrak{a}^\mu R_{00}$	$P_7 \mathfrak{a}^\mu R_{00}$
$u^a R_{ab} \omega^{b\mu}$	$P_8 u^a R_{ab} \omega^{b\mu}$
$\mathfrak{a}_\nu R^{\mu\nu}$	$P_9 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \mathfrak{a}_\nu$
$\mathfrak{a}_\nu F^{\mu\nu}$	$P_{10} \left(F^{\mu\nu} \mathfrak{a}_\nu - R_{00} \mathfrak{a}^\mu + u^\mu u_a \mathfrak{a}_b R^{ab} \right)$
$u_a R^{a\mu bc} \omega_{bc}$	$P_{11} \left(u_a R^{a\mu\alpha\beta} \omega_{\alpha\beta} + 2 \omega^{\mu\alpha} u^a R_{a\alpha} \right)$
$u_a \Theta R^{a\mu}$	$P_{12} u_a \Theta R^{a\mu}$
$u^a R_{ab} \sigma^{b\mu}$	$P_{13} u^a R_{ab} \sigma^{b\mu}$
$u_a R^{ab\mu c} \sigma_{bc}$	$P_{14} u_a R^{ab\mu c} \sigma_{bc}$

- First we shall determine the relevant part of the divergence of the terms with the coefficients P_i , $i = 1, 2, 3, 4$ and $12, 13, 14$. These are easy to calculate

$$\begin{aligned}
\nabla_\mu [P_1 (F_{ab} \sigma^{ab}) u^\mu] &\Rightarrow P_1 F^{ab} (u \cdot \nabla) \sigma_{ab} \\
\nabla_\mu [P_2 (R_{ab} \sigma^{ab}) u^\mu] &\Rightarrow P_2 R^{ab} (u \cdot \nabla) \sigma_{ab} \\
\nabla_\mu [P_3 u^\mu \Theta R] &\Rightarrow P_3 R (u \cdot \nabla) \Theta \\
\nabla_\mu [P_4 u^\mu \Theta R_{00}] &\Rightarrow P_4 R_{00} (u \cdot \nabla) \Theta
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\nabla_\mu [P_{12} P_b^\mu u_a \Theta R^{ab}] &\Rightarrow P_{12} u_a R^{ab} P_b^\mu \nabla_\mu \Theta \\
\nabla_\mu [P_{13} u^a R_{ab} \sigma^{b\mu}] &\Rightarrow P_{13} u^a R_{ab} P_\nu^b \nabla_\mu \sigma^{\mu\nu} \\
\nabla_\mu [P_{14} u_a P_\nu^\mu R^{ab\mu c} \sigma_{bc}] &\Rightarrow P_{14} u_a P_\nu^\mu R^{ab\mu c} \nabla_\mu \sigma_{bc}
\end{aligned}$$

Now we shall compute the divergence of the difficult terms multiplying the coefficients P_9 , P_{10} and P_{11} respectively. We first analyse the situation where in a given basis all the fluid data are locally zero upto the required order and only the curvature data are turned on. For such configurations it will turn out that only these three

terms can produce non-zero divergence. They are given by the following expressions⁴.

$$\nabla_\mu \left[P_9 \left(R^{\mu b} - \frac{1}{2} g^{\mu b} R \right) \mathbf{a}_b \right] = P_9 \left[-\frac{R.R_{00}}{2} + R^{ab} F_{ab} \right] \quad (4.18)$$

$$\nabla_\mu \left[P_{10} \left(F^{\mu\nu} \mathbf{a}_\nu - R_{00} \mathbf{a}^\mu + u^\mu u_a \mathbf{a}_b R^{ab} \right) \right] = P_{10} \left[F^{ab} F_{ab} - R_{00}^2 \right] \quad (4.19)$$

$$\begin{aligned} & \nabla_\mu \left[P_{11} \left(u_a R^{a\mu\alpha\beta} \omega_{\alpha\beta} + 2\omega^{\mu\alpha} u^a R_{a\alpha} \right) \right] \\ &= P_{11} \left[2u^a u^b R_{ac} P^{cd} R_{db} + A^{\mu\nu\lambda} \left(\frac{1}{2} A_{\mu\nu\lambda} + A_{\lambda\nu\mu} \right) \right] \end{aligned} \quad (4.20)$$

where $A^{\mu\nu\lambda} = u_\rho R^{\rho abc} P_a^\mu P_b^\nu P_c^\lambda$.

Now from these three equations (4.18), (4.19) and (4.20) we can conclude the following.

- The last term in the RHS of (4.20) contains only the (*pseudotensor*)² and a (*vector*)², but cannot produce any term proportional to R_{00}^2 or $F_{ab} F^{ab}$. Therefore to have positivity of the divergence for all values of F^2 and R_{00}^2 we must set P_{10} to zero.
- Once P_{10} is set to zero, there are no terms in the final expressions of divergence that contain $(R_{\langle\mu\nu\rangle})^2$, $(F_{\langle\mu\nu\rangle})^2$, R^2 or R_{00}^2 . Therefore in the full divergence the coefficients of all the terms linear in these four independent data must be zero. To satisfy this condition we have to set the following coefficients to zero.
 1. $P_1 = 0$ as it is the total coefficient of the term $F^{ab}(u.\nabla)\sigma_{ab}$.
 2. $P_2 = 0$ as it is the total coefficient of the term $R^{ab}(u.\nabla)\sigma_{ab}$.
 3. $P_3 = 0$ as it is the total coefficient of the term $R(u.\nabla)\Theta$.
 4. $Q_{10} + P_4 = 0$ as it is the total coefficient of the term $R_{00}(u.\nabla)\Theta$.
 5. $P_9 = 0$ as it is the total coefficient of the term $R.R_{00}$.

⁴In this computation, apart from the explicit curvature there is one more source for producing the curvature terms. These are arising because we want to write the final answer for the fluid data in a given basis as chosen in section 3. For example while computing the left hand side of equation (4.18), we shall get a term like $R(\nabla.\mathbf{a})$. However, our basis of independent second order fluid data contains a single scalar $(u.\nabla)\Theta$. Therefore we have to express $(\nabla.\mathbf{a})$ in terms $(u.\nabla)\Theta$ before setting the fluid data to zero. In this process we shall generate a curvature term R_{00} as calculated in equation (A.3). Similar techniques have been used to compute the divergence of the other two terms multiplying P_{10} and P_{11} .

$C_{1,1,1}$ type data

In the fourth order divergence we are not interested in any terms that are multiplied by Θ or $\sigma_{\mu\nu}$. Therefore in the third order entropy current we did not need to consider the $C_{1,1,1}$ type terms which contains more than one factor of Θ , $\sigma_{\mu\nu}$ or both.

Here we are listing only those terms which we shall require for our analysis.

Table 15. $C_{1,1,1}$ type data (Only the relevant ones)

Scalars	Vectors
$K_1 u^\mu \omega^2 \Theta$	$K_5 \omega^{\mu a} \sigma_{ab} \mathbf{a}^b$
$K_2 u^\mu \mathbf{a}^2 \Theta$	$K_6 \sigma^{\mu a} \omega_{ab} \mathbf{a}^b$
$K_3 u^\mu (\mathbf{a}_b \mathbf{a}_c \sigma^{bc})$	$K_7 \omega^{\mu a} \mathbf{a}_b \Theta$
$K_4 u^\mu (\omega_{ab} \sigma_c^b \omega^{ca})$	$K_8 \omega^2 \mathbf{a}^\mu$
	$K_9 \mathbf{a}^2 \mathbf{a}^\mu$
	$K_{10} \omega^{\mu a} \omega_{ab} \mathbf{a}^b$

Now we shall compute relevant part in the divergence of the each of the relevant term. It will turn out that analysing the explicit expression of the divergence we can further set some of the non-zero coefficients to zero.

- Here also the relevant parts are easy to calculate for the terms with the coefficients K_i , $i = 1, 2, 3, 4, 5, 6$ and 7 .

$$\begin{aligned}
\nabla_\mu [K_1 u^\mu \omega^2 \Theta] &\Rightarrow K_1 \omega^2 (u \cdot \nabla) \Theta \\
\nabla_\mu [K_2 u^\mu \mathbf{a}^2 \Theta] &\Rightarrow K_2 \mathbf{a}^2 (u \cdot \nabla) \Theta \\
\nabla_\mu [K_3 u^\mu (\mathbf{a}_b \mathbf{a}_c \sigma^{bc})] &\Rightarrow K_3 \mathbf{a}_b \mathbf{a}_c (u \cdot \nabla) \sigma^{bc} \\
\nabla_\mu [K_4 u^\mu (\omega_{ab} \sigma_c^b \omega^{ca})] &\Rightarrow K_4 \omega_{ab} [(u \cdot \nabla) \sigma_c^b] \omega^{ca}
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
\nabla_\mu [K_5 \omega^{\mu a} \sigma_{ab} \mathbf{a}^b] &\Rightarrow K_5 \omega^{\mu a} [\nabla_\mu \sigma_{ab}] \mathbf{a}^b \\
\nabla_\mu [K_6 \sigma^{\mu a} \omega_{ab} \mathbf{a}^b] &\Rightarrow K_6 [\nabla_\mu \sigma^{\mu a}] \omega_{ab} \mathbf{a}^b \\
\nabla_\mu [K_7 \omega^{\mu b} \mathbf{a}^b \Theta] &\Rightarrow K_7 [\omega^{\mu b} \mathbf{a}^b \nabla_\mu \Theta]
\end{aligned}$$

- The divergence of the terms with coefficients K_8 , K_9 and K_{10} are complicated. These are given by the following expressions.

‘ K_8 -term’

$$\begin{aligned}
& \nabla_\mu [K_8 \omega^2 \mathbf{a}^\mu] \\
&= [(\mathbf{a} \cdot \nabla) K_8] \omega^2 + 2K_8 \omega^{\mu\nu} (\mathbf{a} \cdot \nabla) \omega_{\nu\mu} + K_8 \omega^2 (\nabla \cdot \mathbf{a}) \\
&\Rightarrow -T \left(\frac{dK_8}{dT} \right) \mathbf{a}^2 \omega^2 + 2K_8 \omega^{\mu\nu} (\mathbf{a} \cdot \nabla) \omega_{\nu\mu} \\
&\quad + K_8 \omega^2 [\omega^2 + (u \cdot \nabla) \Theta + R_{00}]
\end{aligned} \tag{4.22}$$

In the last step we have kept only the relevant terms and used (A.1) and (A.3) for simplification.

K_9 -term

$$\begin{aligned}
& \nabla_\mu [K_9 \mathbf{a}^2 \mathbf{a}^\mu] \\
&= \mathbf{a}^2 (\mathbf{a} \cdot \nabla) K_9 + K_9 \mathbf{a}^2 (\nabla \cdot \mathbf{a}) + 2K_9 \mathbf{a}^\mu \mathbf{a}^\nu \nabla_\mu \mathbf{a}_\nu \\
&\Rightarrow - \left(T \frac{dK_9}{dT} + K_9 \right) \mathbf{a}^4 + K_9 \mathbf{a}^2 \left[\omega^2 + \frac{5}{3} (u \cdot \nabla) \Theta + R_{00} \right] \\
&\quad + 2K_9 \mathbf{a}^\mu \mathbf{a}^\nu [(u \cdot \nabla) \sigma_{\mu\nu} + F_{\mu\nu} + \omega_{\mu a} \omega^a{}_\nu]
\end{aligned} \tag{4.23}$$

In the last step we have used equations (A.1), (A.3) and (A.7).

K_{10} -term

$$\begin{aligned}
& \nabla_\mu [K_{10} \omega^{\mu a} \omega_{ab} \mathbf{a}^b] \\
&= K_{10} \left[(\nabla_\mu \omega^{\mu a}) \omega_{ab} \mathbf{a}^b + \omega^{\mu a} \mathbf{a}^b (\nabla_\mu \omega_{ab}) + \omega^{\mu a} \omega_{ab} (\nabla_\mu \mathbf{a}^b) \right] \\
&\quad + (\nabla_\mu K_{10}) \omega^{\mu a} \omega_{ab} \mathbf{a}^b \\
&\Rightarrow - \left(T \frac{dK_{10}}{dT} + K_{10} \right) (\mathbf{a}_\mu \omega^{\mu a} \omega_{ab} \mathbf{a}^b) + K_{10} \omega^{\mu a} \mathbf{a}^b (\nabla_\mu \omega_{ab}) \\
&\quad + K_{10} \omega^\mu{}_a \omega^{a\nu} [(u \cdot \nabla) \sigma_{\mu\nu} + F_{\mu\nu} + \omega_{\mu b} \omega^b{}_\nu] \\
&\quad - K_{10} \omega_{\mu b} \mathbf{a}^b \left[-\nabla_a \sigma^{a\mu} + \frac{2}{3} \nabla^\mu \Theta + u_a R^{a\mu} + \mathbf{a}_b \omega^{b\mu} \right]
\end{aligned} \tag{4.24}$$

In the last step we have used relevant part of equations (A.1), (A.6) and (A.8).

Now as explained before, all the terms that are linear in R_{00} , R , R_{ij} and F_{ij} should be set to zero. This will imply the following for the $C_{1,1,1}$ part of the entropy current.

1. $K_8 = 0$ as it is the total coefficient of the term $\omega^2 R_{00}$.
2. $K_9 = 0$ as it is the total coefficient of the term $\mathbf{a} \cdot F \cdot \mathbf{a}$
3. $K_{10} = 0$ as it is the total coefficient of the term $Tr[\omega \cdot F \cdot \omega]$

- Once K_8 , K_9 and K_{10} are zero there are no terms in the fourth order divergence which are proportional to $[\omega^2]^2$, $[\mathbf{a}^2]^2$, $\mathbf{a}^2\omega^2$ or $[\mathbf{a}.\omega]^2$. This will imply that in the divergence the net coefficients of all the terms, linear in \mathbf{a}^2 , ω^2 , $\omega_{\mu a}\omega^{a\nu}$ or $\mathbf{a}_\mu\omega^{\mu\nu}$ should be zero.

To satisfy this condition we have to set all the K_i from $i = 1, \dots, 5$ to zero. This will also set Q_{11} to zero.

K_6 and K_7 get related to B_1 and B_5 in the following way (see (B.4) and (B.5)).

$$\frac{K_6}{\eta} = \frac{K_7}{\zeta} = \frac{1}{s} \left(\frac{dB_5}{dT} + \frac{2B_1}{T} + 2\frac{dB_1}{dT} \right)$$

Absence of these four fourth order terms mentioned above will also impose some constraints on the transport coefficients of second order stress-tensor by requiring that the coefficients of the four terms $\Theta\mathbf{a}^2$, $\Theta\omega^2$, $\omega_{ab}\sigma_c^b\omega^{ca}$ and $\mathbf{a}_a\sigma^{ab}\mathbf{a}_b$ in the third order divergence should vanish.

5 Constraints on 2nd order transport coefficients

In this section we shall finally analyse how this condition of local entropy production constrains the second order transport coefficients. In the first part of this section we shall derive these constraints. These include the set of five relations among the 15 transport coefficients (as mentioned in the section 1) and also two inequalities involving both the first and second order transport coefficients as well as some coefficients appearing in the entropy current.

Then in the next subsection we shall compare our final result with the answer presented in [2] and [1].

5.1 Derivation of the constraints

At second order, just from symmetry analysis, the stress tensor will have 15 transport coefficients.

$$\begin{aligned} \Pi_{\mu\nu} = & T \left[\tau (u.\nabla)\sigma_{\langle\mu\nu\rangle} + \kappa_1 R_{\langle\mu\nu\rangle} + \kappa_2 F_{\langle\mu\nu\rangle} + \lambda_0 \Theta \sigma_{\mu\nu} \right. \\ & \left. + \lambda_1 \sigma_{\langle\mu}^a \sigma_{a\nu\rangle} + \lambda_2 \sigma_{\langle\mu}^a \omega_{a\nu\rangle} + \lambda_3 \omega_{\langle\mu}^a \omega_{a\nu\rangle} + \lambda_4 \mathbf{a}_{\langle\mu} \mathbf{a}_{\nu\rangle} \right] \\ & + TP_{\mu\nu} \left[\zeta_1 (u.\nabla)\Theta + \zeta_2 R + \zeta_3 R_{00} + \xi_1 \Theta^2 + \xi_2 \sigma^2 + \xi_3 \omega^2 + \xi_4 \mathbf{a}^2 \right] \end{aligned} \quad (5.1)$$

As explained before, in the expression of the divergence of the entropy current, $\Pi^{\mu\nu}$ will always appear contracted with $\sigma_{\mu\nu}$ and Π with Θ . Therefore, in Π^{ab} all the terms, which have either σ_{ab} or Θ as factors, will finally generate a set of quadratic and higher order terms in σ_{ab} and Θ . Such terms are always suppressed in derivative

expansion over the second order piece of the divergence provided the shear and the bulk viscosities are non zero. Therefore the coefficients multiplying these terms can never be constrained from the condition of positivity. Among the 15 transport coefficients, five (λ_0 , λ_1 , λ_2 , ξ_1 and ξ_2) are of such type and therefore are completely unconstrained.

It turns out that to maintain the positivity of the divergence, the coefficients τ and ζ_1 have to satisfy some inequalities. This is because at fourth order, the divergence of the entropy current will contain terms proportional to $[(u.\nabla)\sigma]^2$ and $[(u.\nabla)\Theta]^2$ whose coefficients are Q_2 and Q_1 respectively (see (4.14)). These two terms, along with four other terms (σ^2 , Θ^2 , $\sigma^{\mu\nu}(u.\nabla)\sigma_{\mu\nu}$ and $\Theta(u.\nabla)\Theta$, appearing in the second and third order pieces of the divergence) together can be made positive definite provided the transport coefficients τ and ζ_1 satisfy the following inequalities.

$$\begin{aligned}(\zeta_1 - C_\Theta)^2 &\leq 4\zeta Q_1 \\ (\tau - C_\sigma)^2 &\leq 4\eta Q_2\end{aligned}\tag{5.2}$$

Where C_Θ and C_σ are the coefficients of the term $\Theta(u.\nabla)\Theta$ and $\sigma^{ab}(u.\nabla)\sigma_{ab}$ respectively in the divergence of the third order entropy current.

$$\begin{aligned}C_\Theta &= 2s\frac{dB_5}{ds} - \frac{2}{3}T\frac{dB_5}{dT} + 2B_2 + 2B_4s\left(s - T\frac{ds}{dT}\right) \\ C_\sigma &= T\frac{dB_5}{dT} + 2B_3\end{aligned}\tag{5.3}$$

But unlike the inequalities for the first order transport coefficients ($\eta \geq 0$ and $\zeta \geq 0$) (5.2) involves several free coefficients appearing in the entropy current and hence it does not give any relation within the transport coefficients themselves.

Now we shall come to those relations which will give some equalities among the remaining eight transport coefficients. By computing the divergence of the entropy current upto fourth order we can see that there are no terms proportional to R^2 , R_{00}^2 , $R_{ab}R^{ab}$, $F_{ab}F^{ab}$, \mathfrak{a}^4 , ω^4 and $(\mathfrak{a}.\omega)^2$. It will imply that the coefficients of the following 8 terms in the divergence of the entropy current have to be zero.

1. $C_F \equiv$ Coefficient of the term $\sigma_{ab}F^{ab}$
2. $C_R \equiv$ Coefficient of the term $\sigma_{ab}R^{ab}$
3. $C_{\mathfrak{a}} \equiv$ Coefficient of the term $\mathfrak{a}^a\mathfrak{a}^b\sigma_{ab}$
4. $C_\omega \equiv$ Coefficient of the term $\omega^{ap}\omega_p{}^b\sigma_{ab}$
5. $Q_F \equiv$ Coefficient of the term ΘR_{00}
6. $Q_R \equiv$ Coefficient of the term ΘR

7. $Q_a \equiv$ Coefficient of the term Θa^2

8. $Q_\omega \equiv$ Coefficient of the term $\Theta \omega^2$

Solving each of the above eight conditions we can express the remaining eight transport coefficients in terms of the coefficients appearing in the divergence of the second order entropy current.

These are given by the following expressions.

$$\begin{aligned}
C_R = 0 &\Rightarrow \kappa_1 = A_3, & C_F = 0 &\Rightarrow \kappa_2 = T \frac{dB_5}{dT} \\
C_\omega = 0 &\Rightarrow \lambda_3 = T \frac{dB_5}{dT} - 4B_1 \\
C_a = 0 &\Rightarrow \lambda_4 = - \left[T^2 \frac{d^2 B_5}{dT^2} + T \frac{dB_5}{dT} + 2B_4 T^2 \left(\frac{ds}{dT} \right)^2 \right] \\
Q_R = 0 &\Rightarrow \zeta_2 = \frac{1}{2} \left[s \frac{dA_3}{ds} - \frac{A_3}{3} \right] \\
Q_F = 0 &\Rightarrow \zeta_3 = s \frac{dA_3}{ds} + \frac{A_3}{3} - \frac{2T}{3} \frac{dB_5}{dT} - 2B_4 T s \frac{ds}{dT} \\
Q_\omega = 0 &\Rightarrow \xi_3 = -2B_4 T s \frac{ds}{dT} + T \frac{dB_5}{dT} \left[\frac{s}{T} \frac{dT}{ds} - \frac{2}{3} \right] - s \frac{dB_1}{ds} \\
&\quad + B_1 \left[\frac{2s}{T} \frac{dT}{ds} - \frac{1}{3} \right] \\
Q_a = 0 &\Rightarrow \xi_4 = T^2 s \frac{ds}{dT} \frac{dB_4}{dT} + B_4 \left[\frac{T^2}{3} \left(\frac{ds}{dT} \right)^2 + 4T s \frac{ds}{dT} + 2T^2 s \frac{d^2 s}{dT^2} \right] \\
&\quad + \frac{2}{3} \left(T \frac{dB_5}{dT} + T^2 \frac{d^2 B_5}{dT^2} \right) \\
&\text{where } \frac{dB_5}{dT} = \frac{A_3}{T} + \frac{dA_3}{dT}
\end{aligned} \tag{5.4}$$

From (5.4) we can see that all these eight coefficients can be determined in terms of three independent coefficients (A_3 , B_1 and B_4) appearing in the third order entropy current. Therefore eliminating the three entropy current coefficients, mentioned above finally we shall get five relations among these eight transport coefficients.

$$\begin{aligned}
\kappa_2 &= \kappa_1 + T \frac{d\kappa_1}{dT}, & \zeta_2 &= \frac{1}{2} \left[s \frac{d\kappa_1}{ds} - \frac{\kappa_1}{3} \right] \\
\zeta_3 &= \left(s \frac{d\kappa_1}{ds} + \frac{\kappa_1}{3} \right) + \left(s \frac{d\kappa_2}{ds} - \frac{2\kappa_2}{3} \right) + \frac{s}{T} \left(\frac{dT}{ds} \right) \lambda_4 \\
\xi_3 &= \frac{3}{4} \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \left(T \frac{d\kappa_2}{dT} + 2\kappa_2 \right) - \frac{3\kappa_2}{4} + \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \lambda_4 \\
&\quad + \frac{1}{4} \left[s \frac{d\lambda_3}{ds} + \frac{\lambda_3}{3} - 2 \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \lambda_3 \right] \\
\xi_4 &= -\frac{\lambda_4}{6} - \frac{s}{T} \left(\frac{dT}{ds} \right) \left(\lambda_4 + \frac{T}{2} \frac{d\lambda_4}{dT} \right) - T \left(\frac{d\kappa_2}{dT} \right) \left(\frac{3s}{2T} \frac{dT}{ds} - \frac{1}{2} \right) \\
&\quad - \frac{Ts}{2} \left(\frac{dT}{ds} \right) \left(\frac{d^2\kappa_2}{dT^2} \right)
\end{aligned} \tag{5.5}$$

5.2 Comparison with [2]

In [2] authors have constructed some examples of non-conformal fluid which can be obtained by dimensional reduction of some higher dimensional conformal theory. The entropy of such non conformal fluid is proportional to $T^{2\sigma-1}$ where 2σ was the dimension of the space-time before the reduction. Since this particular nonconformal fluid satisfies the condition of ‘positivity’ of the divergence of the entropy current by construction the transport coefficients should also obey the relations listed in (5.5).

Below we are quoting the values of some transport coefficients for such non-conformal fluids. These are the transport coefficients which enter the 5 relations in (5.5).

$$\begin{aligned}
\lambda_3 &= \Lambda_3 T^{2\sigma-3}, & \xi_3 &= \frac{2\sigma-4}{3(2\sigma-1)} \lambda_3 \\
\kappa_1 &= \kappa T^{2\sigma-3}, & \zeta_2 &= \frac{2\sigma-4}{3(2\sigma-1)} \kappa_1 \\
\kappa_2 &= (2\sigma-2)\kappa_1, & \zeta_3 &= \frac{2\sigma-4}{3(2\sigma-2)} \kappa_2 \\
\lambda_4 &= 0, & \xi_4 &= 0
\end{aligned} \tag{5.6}$$

where Λ_3 and κ are two dimensionful constants which depend on the length of the compactified dimensions but independent of temperature. Using the fact that for such dimensionally reduced nonconformal fluids the entropy can be written as

$$s \propto T^{2\sigma-1}$$

one can check that these values satisfy the relations given in (5.5).

5.3 Comparison with [1]

To compare, first we shall express the eight relevant transport coefficients (the ones which appear in equation (5.5)) in terms of the coefficients as given in [1]. The dictionary is the following.

$$\begin{aligned}
T\zeta_2 &= \xi_5^{Rom}, \quad T\zeta_3 = \xi_6^{Rom}, \quad T\xi_3 = -\xi_3^{Rom} \\
T\lambda_3 &= -\lambda_3^{Rom}, \quad T\kappa_1 = \kappa^{Rom}, \quad T\kappa_2 = 2(\kappa - \kappa^*)^{Rom} \\
T\xi_4 &= \frac{T^2}{s^2} \left(\frac{ds}{dT} \right)^2 \xi_4^{Rom}, \quad T\lambda_4 = \frac{T^2}{s^2} \left(\frac{ds}{dT} \right)^2 \lambda_4^{Rom}
\end{aligned} \tag{5.7}$$

The author of [1] has argued for the existence of two relations among 5 of these 8 nondissipative transport coefficients. These two relations are not explicitly presented in the paper [1] in generality. However the author of [1] appears to claim, in the unnumbered equation in section 5 (below equation 32) of [1], that in the special case, when

$$T = s c_s^2, \quad \text{and} \quad \kappa^{Rom} \propto \frac{s}{T} \tag{5.8}$$

the two relations among the five transport coefficients reduce to the following.

$$\begin{aligned}
\xi_5^{Rom} &= \frac{\kappa^{Rom}}{3} [1 - 3c_s^2] \\
\xi_6^{Rom} + \xi_3^{Rom} &= - \left(\frac{3c_s^2 - 1}{3c_s^2} \right) [\kappa^{Rom} + c_s^2 \lambda_3^{Rom}]
\end{aligned} \tag{5.9}$$

The first equation in (5.9) indeed reduces to the second equation in (5.5) when c_s^2 is a constant. In order to compare our results with the second of (5.9), we subtract the fourth equation from the third equation of (5.5) and then use the first equation of (5.5).

This gives a relationship between all the same transport coefficients that appear in the second equation of (5.9). However the relationship we find is the following.

$$\begin{aligned}
\zeta_3 - \xi_3 &= \xi_6^{Rom} + \xi_3^{Rom} \\
&= \left(s \frac{d\kappa_1}{ds} + \frac{\kappa_1}{3} \right) + \frac{1}{4} \left(s \frac{d\kappa_2}{ds} + \frac{\kappa_2}{3} \right) - \frac{3}{2} \frac{s}{T} \frac{dT}{ds} \kappa_2 \\
&\quad - \frac{1}{4} \left[s \frac{d\lambda_3}{ds} + \frac{\lambda_3}{3} - 2 \left(\frac{s}{T} \right) \left(\frac{dT}{ds} \right) \lambda_3 \right]
\end{aligned} \tag{5.10}$$

where

$$\kappa_2 = \kappa_1 + T \frac{d\kappa_1}{dT}$$

This relationship does not reduce to the second of (5.9) after substituting the special case of (5.8) with constant c_s . We do not understand the reason for this disagreement. Perhaps the second of (5.9) applies under more restrictive assumptions than stated explicitly in [1]. As noted in [1] it certainly applies to the particular case, described in [2].

6 Conformal limit

Upto second order in derivative expansion the final entropy current (consistent with the constraint of non-negative divergence) is given by the following expression

$$\begin{aligned} & \tilde{J}^\mu|_{\text{second order}} \\ &= \nabla_\nu [A_1(u^\mu \nabla^\nu T - u^\nu \nabla^\mu T)] + \nabla_\nu (A_2 T \omega^{\mu\nu}) \\ &+ A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu + \left(\frac{A_3}{T} + \frac{dA_3}{dT} \right) [\Theta \nabla^\mu T - P^{ab} (\nabla_b u^\mu) (\nabla_a T)] \\ &+ (B_1 \omega^2 + B_2 \Theta^2 + B_3 \sigma^2) u^\mu + B_4 [(\nabla s)^2 u^\mu + 2s \Theta \nabla^\mu s] \end{aligned} \quad (6.1)$$

If the theory has conformal symmetry, then the entropy current also should transform covariantly under a conformal transformation. The conformally covariant entropy current is a special case of equation (6.1). In this case the only available length scale is provided by the temperature and therefore the temperature dependence of all the coefficients are fixed just by dimensional argument and also some of the coefficients are related to the others in a way so that the terms that transform in-homogeneously under conformal transformation cancel.

At second order in derivative expansion, there are three scalars and two vectors [1, 5, 8] which transform covariantly under conformal transformation. In our basis these are given by the following combinations

$$\begin{aligned} \mathcal{S}_1 &= \sigma_{ab} \sigma^{ba}, & \mathcal{S}_2 &= \omega_{ab} \omega^{ba} \\ \mathcal{S}_3 &= \frac{P^{ab} \nabla_a \nabla_b T}{T} - \frac{P^{ab} (\nabla_a T) (\nabla_b T)}{2T^2} - \frac{R_{00}}{2} - \frac{R}{4} + \frac{\Theta^2}{6} \\ \mathcal{V}_1^\mu &= P_a^\nu \nabla_\mu \sigma^{\mu a} - 3\mathfrak{a}_\mu \sigma^{\mu\nu} & \mathcal{V}_2^\mu &= P_a^\nu \nabla_\mu \omega^{\mu a} - \mathfrak{a}_\mu \omega^{\mu\nu} \end{aligned} \quad (6.2)$$

A conformally covariant entropy current should be expressible only in terms of these three scalars and two vectors. So to begin with it can have five independent coefficients. Then the constraint of positivity will reduce it to some special case of (6.1).

Here we have used (6.1) to deduce the conformally covariant form of the entropy current. First we have fixed the temperature dependence of the coefficients A_i and B_i by dimensional analysis. Then we have tried to figure out the minimal set of relations these coefficients have to satisfy such that all the terms transforming in-homogeneously under conformal transformation cancel. This means that one should

be able to choose the coefficients A_i and B_i in such a way so that the entropy current is expressible in terms of these 3 conformal scalars and 2 conformal vectors. To do this we first rearrange some of the terms appearing in equation (6.1) assuming that the temperature dependence of the coefficients are fixed by dimensional analysis.

$$\begin{aligned} & \nabla_\nu [A_1(u^\mu \nabla^\nu T - u^\nu \nabla^\mu T)] \\ &= A_1 \left[u^\mu \mathcal{S}_3 - \frac{1}{2} (\mathcal{V}_1^\mu + \mathcal{V}_2^\mu) + \frac{u^\mu}{2} \left(\mathfrak{a}^2 - \Theta^2 + R_{00} + \frac{R}{2} \right) \right. \\ & \quad \left. - \mathfrak{a}_b (\sigma^{b\mu} + \omega^{b\mu}) + \frac{\Theta}{3} \mathfrak{a}^\mu - \frac{1}{2} u^k R_{ka} P^{\mu a} \right] \end{aligned} \quad (6.3)$$

$$\begin{aligned} & \nabla_\mu [A_2 \omega^{\mu\nu}] = A_2 [\mathcal{V}_2^\nu - \mathcal{S}_2 u^\nu] \\ & A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu = A_3 \left[-u^\mu \left(\frac{R}{2} + R_{00} \right) + P^{\mu a} R_{ab} u^b \right] \end{aligned} \quad (6.4)$$

$$\begin{aligned} & \left(\frac{A_3}{T} + \frac{dA_3}{dT} \right) [\Theta \nabla^\mu T - P^{ab} (\nabla_b u^\mu) (\nabla_a T)] \\ &= 2A_3 \left[\frac{\Theta^2}{3} u^\mu - \frac{2\Theta}{3} \mathfrak{a}^\mu + \mathfrak{a}_b (\sigma^{b\mu} + \omega^{b\mu}) \right] \end{aligned} \quad (6.5)$$

$$B_4 [(\nabla s)^2 u^\mu + 2s \Theta \nabla^\mu s] = T^6 B_4 \left[\Theta^2 u^\mu + 9 \left(\mathfrak{a}^2 u^\mu - \frac{2\Theta}{3} \mathfrak{a}^\mu \right) \right] \quad (6.6)$$

From these expressions we can see how one should choose the coefficients A_i and B_i such that all the pieces that transform inhomogeneously under conformal transformation cancel. The coefficients for a conformally covariant entropy current are given by the following expressions.

$$\begin{aligned} & A_1(T) = a_1, \quad A_2(T) = a_2, \quad A_3(T) = \frac{a_1}{2} T \\ & B_1(T) = b_1 T, \quad B_2(T) = \frac{2a_1}{9} T, \quad B_3(T) = b_3 T \\ & B_4(T) = - \left(\frac{a_1}{18} \right) T^{-5} \end{aligned} \quad (6.7)$$

where all a_i and b_i are constants.

Therefore the conformally covariant entropy current has four independent coefficients (a_1 , a_2 , b_1 and b_3) when expanded upto second order in derivatives. When written in terms of these four coefficients the expressions for the conformal entropy current is given as

$$\begin{aligned} & J_{\text{conformal}}^\mu \\ &= a_1 T \mathcal{S}_3 u^\mu + \frac{a_1 T}{2} (\mathcal{V}_1^\mu + \mathcal{V}_2^\mu) + a_2 T [\mathcal{V}_2^\nu - \mathcal{S}_2 u^\nu] + b_1 T \mathcal{S}_2 u^\mu + b_2 T \mathcal{S}_1 u^\mu \\ &= T [a_1 \mathcal{S}_3 + b_2 \mathcal{S}_1 + (b_1 - a_2) \mathcal{S}_2] u^\mu + T \left(a_2 + \frac{a_1}{2} \right) \mathcal{V}_2^\mu + \frac{a_1 T}{2} \mathcal{V}_1^\mu \end{aligned} \quad (6.8)$$

This expression coincides with the expression presented in [8] and [1] with the following identification.

$$\begin{aligned}
a_1 T &= 4A_3^{Rom} \\
b_2 T &= \frac{A_1^{Rom}}{4} - \frac{A_3^{Rom}}{2} + \frac{B_1^{Rom}}{4} \\
T(b_1 - a_2) &= A_2^{Rom} + 2A_3^{Rom} - B_2^{Rom} \\
\frac{a_1 T}{2} &= B_1^{Rom} \\
T\left(a_2 + \frac{a_1}{2}\right) &= B_2^{Rom}
\end{aligned} \tag{6.9}$$

where A_i^{Rom} and B_i^{Rom} are the coefficients in the conformal entropy current as defined in [1].

Substituting the relations (6.7) in (5.4) one can see that in conformal case ζ_2 , ζ_3 , ξ_3 , ξ_4 and λ_4 vanish and κ_2 is related to κ_1 as

$$\kappa_2 = 2\kappa_1$$

However once the stress tensor is conformally covariant, all these vanishing of the coefficients and the relation between κ_1 and κ_2 are automatic (If these relations were not true then the stress tensor would have some terms which will transform in-homogeneously under conformal transformation). Therefore we can say that the existence of an entropy with positive divergence does not constrain the uncharged conformal fluid.

7 Acknowledgement

I would like to thank Shiraz Minwalla for suggesting this problem, collaborating in the initial part of the calculation and providing guidance at every stage. I would like to thank T. Sharma and S. Jain for rechecking the calculation presented in section 3. I would also like to thank N. Banerjee, S. Jain, T. Sharma and everyone in HRI for useful discussion. Finally I would like to acknowledge our debt to the people of India for their generous and steady support to research in the basic science.

8 Appendices

A Identities

Here we list the identities that we have used to calculate the divergence and then to transform the answer to the required basis.

$$\begin{aligned}
(u \cdot \nabla)s + s\Theta &= \frac{\eta\sigma^2 + \zeta\Theta^2}{T} + \dots \\
P^{\mu\nu}\nabla_\nu T + T\mathfrak{a}^\mu &= \frac{P_a^\mu\nabla_\nu[\eta\sigma^{\nu a} + \zeta\Theta P^{\nu a}]}{s} + \dots
\end{aligned} \tag{A.1}$$

where the RHS of the second equation can be further simplified.

$$\begin{aligned} & \frac{P_a^\mu \nabla_\nu [\eta \sigma^{\nu a} + \zeta \Theta P^{\nu a}]}{s} \\ &= \frac{1}{s} \left[-T \frac{d\eta}{dT} \mathbf{a}_b \sigma^{b\mu} + \left(\zeta - T \frac{d\zeta}{dT} \right) \mathbf{a}^\mu \Theta + \eta P_a^\mu \nabla_b \sigma^{ab} + \zeta P^{\mu a} \nabla_a \Theta \right] \end{aligned} \quad (\text{A.2})$$

$$(\nabla \cdot \mathbf{a}) = \left(\sigma^2 + \omega^2 + \frac{\Theta^2}{3} \right) + (u \cdot \nabla) \Theta + R_{00} \quad (\text{A.3})$$

$$\begin{aligned} (u \cdot \nabla) \mathbf{a}_\nu &= \mathbf{a}_\nu \Theta \left[2 \frac{s}{T} \frac{dT}{ds} - \frac{4}{3} - s \frac{ds}{dT} \frac{d^2 T}{ds^2} \right] - \mathbf{a}^b \sigma_{\nu b} \\ &+ \left[u_\nu \mathbf{a}^2 - \mathbf{a}^b \omega_{\nu b} + \frac{s}{T} \frac{dT}{ds} P_\nu^\alpha \nabla_\alpha \Theta \right] + \dots \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} P_{\nu\mu} \nabla_a \omega^{a\nu} &= P_{\nu\mu} \nabla_a \sigma^{a\nu} - \frac{2}{3} P_{\nu\mu} \nabla^\nu \Theta - P_{\mu\nu} u_a R^{a\nu} \\ &- \mathbf{a}^b (\sigma_{b\mu} + \omega_{b\mu}) \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \omega^{\nu\mu} (u \cdot \nabla) \omega_{\mu\nu} &= -2 \omega_{ab} \sigma_c^b \omega^{ca} - \omega^2 \left(\frac{2\Theta}{3} + \frac{u \cdot \nabla T}{T} \right) \\ &+ \frac{1}{Ts} \left[1 + \frac{T}{s} \frac{ds}{dT} \right] \mathbf{a}_\mu \omega^{\mu\nu} (\nabla_a \Pi_\nu^a) + \frac{\omega^{\mu\nu} \nabla_\mu (\nabla_a \Pi_\nu^a)}{Ts} \\ &+ \frac{\omega^2}{Ts} \left(\sigma_{\mu\nu} \Pi^{\mu\nu} + \frac{\Theta}{3} \Pi \right) + \dots \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} \mathbf{a}^\mu \mathbf{a}^\nu (\nabla_\mu \mathbf{a}_\nu) &= \mathbf{a}^\mu \mathbf{a}^\nu \left[\sigma_\mu^a \sigma_{a\nu} + \omega_{\mu a} \omega^a{}_\nu + \frac{2\Theta}{3} \sigma_{\mu\nu} \right] \\ &+ \mathbf{a}^\mu \mathbf{a}^\nu \left[(u \cdot \nabla) \sigma_{\mu\nu} + F_{\mu\nu} + \frac{P_{\mu\nu}}{3} (u \cdot \nabla) \Theta \right] \\ &+ \frac{\mathbf{a}^2 \Theta^2}{9} - (\mathbf{a}^2)^2 \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} \omega^\mu{}_a \omega^{a\nu} (\nabla_\mu \mathbf{a}_\nu) &= \omega^\mu{}_a \omega^{a\nu} \left[\sigma_\mu^b \sigma_{b\nu} + \omega_{\mu b} \omega^b{}_\nu + \frac{2\Theta}{3} \sigma_{\mu\nu} \right] \\ &+ \omega^\mu{}_a \omega^{a\nu} \left[(u \cdot \nabla) \sigma_{\mu\nu} + F_{\mu\nu} + \frac{P_{\mu\nu}}{3} (u \cdot \nabla) \Theta \right] \\ &+ \frac{\omega^2 \Theta^2}{9} - \mathbf{a}_\mu \mathbf{a}_\nu \omega^\mu{}_a \omega^{a\nu} \end{aligned} \quad (\text{A.8})$$

$$\sigma^{\mu\nu}(\nabla_\mu \mathbf{a}_\nu) = \sigma^{\mu\nu} \left[\sigma_\mu^a \sigma_{a\nu} + \omega_\mu^a \omega_{a\nu} + F_{\mu\nu} + (u \cdot \nabla) \sigma_{\mu\nu} - \mathbf{a}_\mu \mathbf{a}_\nu \right] + \frac{2\Theta}{3} \sigma^2 \quad (\text{A.9})$$

$$2\mathbf{a}^\alpha \omega^{\mu\nu} \nabla_\nu \omega_{\mu\alpha} = -\omega^{\mu\nu} (\mathbf{a} \cdot \nabla) \omega_{\nu\mu} + \omega^2 \mathbf{a}^2 - \mathbf{a}_\mu \omega^{\mu\nu} \omega_{\nu\alpha} \mathbf{a}^\alpha \quad (\text{A.10})$$

The identity (A.10) is derived using the following steps.

$$\begin{aligned} & \omega^{\mu\nu} \nabla_\nu [\nabla_\mu u_\alpha - \nabla_\alpha u_\mu] \\ &= \omega^{\mu\nu} \nabla_\nu \left(2\omega_{\mu\alpha} - u_\mu \mathbf{a}_\alpha + u_\alpha \mathbf{a}_\mu \right) \\ &= 2\omega^{\mu\nu} \nabla_\nu \omega_{\mu\alpha} - \omega^2 \mathbf{a}_\alpha + \omega^{\mu\nu} (\mathbf{a}_\mu \nabla_\nu u_\alpha + u_\alpha \nabla_\nu \mathbf{a}_\mu) \\ &= \omega^{\mu\nu} \left(\frac{1}{2} [\nabla_\nu, \nabla_\mu] u_\alpha - \nabla_\alpha (\nabla_\nu u_\mu) - [\nabla_\nu, \nabla_\alpha] u_\mu \right) \\ &= \omega^{\mu\nu} \left(-\nabla_\alpha \omega_{\nu\mu} + \mathbf{a}_\mu \omega^{\mu\nu} \nabla_\alpha u_\nu + u^\rho \left[\frac{1}{2} R_{\rho\alpha\mu\nu} - R_{\rho\mu\alpha\nu} \right] \right) \\ &= \omega^{\mu\nu} \left(-\nabla_\alpha \omega_{\nu\mu} + \mathbf{a}_\mu \omega^{\mu\nu} \nabla_\alpha u_\nu - \frac{u^\rho}{2} [R_{\rho\nu\alpha\mu} + R_{\rho\mu\alpha\nu}] \right) \\ &= \omega^{\mu\nu} \left(-\nabla_\alpha \omega_{\nu\mu} + \mathbf{a}_\mu \omega^{\mu\nu} \nabla_\alpha u_\nu \right) \end{aligned}$$

$$\begin{aligned} 2A^{\mu\nu\lambda} \nabla_\nu \sigma_{\lambda\mu} &= A^{\mu\nu\lambda} \left[\nabla_\mu \omega_{\nu\lambda} + \frac{1}{2} A_{\mu\lambda\nu} + A_{\lambda\nu\mu} \right. \\ &\quad \left. - \omega_{\nu\lambda} \mathbf{a}_\nu - 2\omega_{\nu\mu} \mathbf{a}_\lambda - \frac{2}{3} (u_\rho R^{\rho b} P_b^\nu) (\mathbf{a}_\nu \Theta) \right] \end{aligned} \quad (\text{A.11})$$

where $A^{\mu\nu\lambda} = u_\rho R^{\rho abc} P_a^\mu P_b^\nu P_c^\lambda$

The identity (A.11) can be derived using the similar tricks as in the identity (A.10).

$$\begin{aligned} & A^{\mu\nu\lambda} \nabla_\nu [\nabla_\lambda u_\mu + \nabla_\mu u_\lambda] \\ &= A^{\mu\nu\lambda} \nabla_\nu \left[2\sigma_{\lambda\mu} + \frac{2}{3} P_{\lambda\mu} \Theta - u_\lambda \mathbf{a}_\mu - u_\mu \mathbf{a}_\lambda \right] \\ &= A^{\mu\nu\lambda} \left(\frac{1}{2} [\nabla_\nu, \nabla_\lambda] u_\mu + [\nabla_\nu, \nabla_\mu] u_\lambda + \nabla_\mu (\nabla_\nu u_\lambda) \right) \\ &= A^{\mu\nu\lambda} \left[\frac{1}{2} A_{\mu\lambda\nu} + A_{\lambda\mu\nu} + \nabla_\mu \omega_{\nu\lambda} - \mathbf{a}_\lambda \nabla_\nu u_\mu \right] \end{aligned} \quad (\text{A.12})$$

B Computation of the divergence

Here we shall calculate the divergence of the different terms appearing in the second order entropy current. The final expression for the second order entropy current is given in (6.1). Here we are quoting the equation again.

$$\begin{aligned}
& \tilde{J}^\mu|_{\text{second order}} \\
&= \nabla_\nu [A_1(u^\mu \nabla^\nu T - u^\nu \nabla^\mu T)] + \nabla_\nu (A_2 T \omega^{\mu\nu}) \\
&\quad + A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu + \left(\frac{A_3}{T} + \frac{dA_3}{dT} \right) [\Theta \nabla^\mu T - P^{ab} (\nabla_b u^\mu) (\nabla_a T)] \\
&\quad + (B_1 \omega^2 + B_2 \Theta^2 + B_3 \sigma^2) u^\mu + B_4 [(\nabla s)^2 u^\mu + 2s \Theta \nabla^\mu s]
\end{aligned} \tag{B.1}$$

The first two terms (with coefficients A_1 and A_2 respectively) have zero divergence. Below we shall calculate the divergence of the rest of the terms. As explained before, to determine the constraints of ‘positivity’ we need to calculate the divergence upto fourth order in derivative expansion. However in the fourth order piece of the divergence we need to retain only those terms which do not involve any factor of $\sigma_{\mu\nu}$ or Θ .

Divergence of the term with coefficient A_3 :

$$\begin{aligned}
& \nabla_\mu \left[A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu \right]_{\text{upto 3rd order}} \\
&= (\nabla_\mu A_3) \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu + A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) (\nabla_\mu u_\nu) \\
&= (\nabla_\mu A_3) R^{\mu\nu} u_\nu - \frac{R}{2} (u \cdot \nabla) A_3 + A_3 \left(R^{\mu\nu} - \frac{R}{2} g^{\mu\nu} \right) \left(\sigma_{\mu\nu} - u_\mu \mathfrak{a}_\nu + P_{\mu\nu} \frac{\Theta}{3} \right) \\
&= \Theta \left[R_{00} \left(s \frac{dA_3}{ds} + \frac{A_3}{3} \right) + \frac{R}{2} \left(s \frac{dA_3}{ds} - \frac{A_3}{3} \right) \right] \\
&\quad - \left(A_3 + T \frac{dA_3}{dT} \right) (u^\mu \mathfrak{a}^\nu R_{\mu\nu}) + A_3 R_{\mu\nu} \sigma^{\mu\nu} \\
&\quad \nabla_\mu \left[A_3 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu \right]_{\text{relevant terms at 4th order}} \\
&= u_\nu P_\mu^a R^{\mu\nu} \left(\frac{1}{s} \frac{dA_3}{dT} \right) [\eta \nabla_b \sigma^{ab} + \zeta \nabla_a \Theta]
\end{aligned} \tag{B.2}$$

Divergence of the term with coefficient $\left(\frac{A_3}{T} + \frac{dA_3}{dT} \right)$:

$$\begin{aligned}
& \nabla_\mu [\Theta \nabla^\mu B_5 - P^{ab} (\nabla_b u^\mu) (\nabla_a B_5)]_{\text{upto 3rd order}} \\
&= s \frac{dB_5}{ds} \Theta \left[(u \cdot \nabla) \Theta + \sigma^2 + \omega^2 + \frac{\Theta^2}{3} \right] + T \frac{dB_5}{dT} \left[\mathfrak{a}_\mu u_\nu R^{\mu\nu} + \mathfrak{a}_\mu \mathfrak{a}_\nu \sigma^{\mu\nu} + \frac{\mathfrak{a}^2 \Theta}{3} \right] \\
&\quad - \sigma^{\mu\nu} \nabla_\mu \nabla_\nu B_5 + \frac{2\Theta}{3} \nabla^2 B_5 - \frac{\Theta}{3} (u \cdot \nabla)^2 B_5
\end{aligned} \tag{B.3}$$

where

$$\frac{dB_5}{dT} = \frac{A_3}{T} + \frac{dA_3}{dT}$$

The three terms in the second line of equation (B.3) can be simplified further.

$$(u.\nabla)^2 B_5 = \left(s^2 \frac{d^2 B_5}{ds^2} + s \frac{dB_5}{ds} \right) \Theta^2 - s \frac{dB_5}{ds} (u.\nabla) \Theta + T \frac{dB_5}{dT} \mathfrak{a}^2$$

$$\begin{aligned} \nabla^2 B_5 &= -\Theta^2 \left[s^2 \frac{d^2 B_5}{ds^2} + \frac{T}{3} \frac{dB_5}{dT} \right] + \left[s \frac{dB_5}{ds} - T \frac{dB_5}{dT} \right] (u.\nabla) \Theta \\ &+ \left(T \frac{dB_5}{dT} + T^2 \frac{d^2 B_5}{dT^2} \right) \mathfrak{a}^2 - T \frac{dB_5}{dT} (\sigma^2 + \omega^2 + R_{00}) \end{aligned}$$

$$\begin{aligned} \sigma^{\mu\nu} \nabla_\mu \nabla_\nu B_5 &= \left(2T \frac{dB_5}{dT} + T^2 \frac{d^2 B_5}{dT^2} \right) (\mathfrak{a}_\mu \sigma^{\mu\nu} \mathfrak{a}_\nu) + \left(s \frac{dB_5}{ds} - \frac{2T}{3} \frac{dB_5}{dT} \right) \sigma^2 \Theta \\ &- T \frac{dB_5}{dT} \sigma_{\mu\nu} [F^{\mu\nu} + (u.\nabla) \sigma^{\mu\nu} + \sigma^{\mu a} \sigma_a^\nu + \omega^{\mu a} \omega_a^\nu] \end{aligned}$$

The relevant part of the fourth order piece in the divergence is given by the following expression.

$$\begin{aligned} &\nabla_\mu \left[\Theta \nabla^\mu B_5 - P^{ab} (\nabla_b u^\mu) (\nabla_a B_5) \right]_{\text{relevant part at 4th order}} \\ &= - \frac{dB_5}{dT} (\mathfrak{a}_\mu \omega^{\mu a} + u_\nu P_{\mu a} R^{\mu\nu}) \left(\frac{\eta P_b^a \nabla_\mu \sigma^{\mu b} + \zeta P^{\mu a} \nabla_\mu \Theta}{s} \right) \end{aligned} \quad (\text{B.4})$$

Divergence of the term with coefficient B_1 :

$$\begin{aligned}
& \nabla_\mu [B_1 \omega^2 u^\mu]_{\text{upto 3rd order}} \\
&= B_1 \omega^2 \Theta + [(u \cdot \nabla) B_1] \omega^2 - 2B_1 \left[\omega_{ab} \sigma_c^b \omega^{ca} + \omega^2 \frac{\Theta}{3} - \frac{s}{2T} \frac{dT}{ds} \omega^2 \Theta \right] \\
&= \left[-s \frac{dB_1}{ds} - \frac{B_1}{3} + 2B_1 \left(\frac{s}{T} \frac{dT}{ds} \right) \right] \omega^2 \Theta - 4B_1 \sigma_\mu^a \omega^{\mu\nu} \omega_{\nu a} \\
& \nabla_\mu \left[B_1 \omega^2 u^\mu + \left(\frac{2B_1}{Ts} \right) \omega^{b\mu} (\nabla_a \Pi_b^a) \right]_{\text{relevant part at 4th order}} \\
&= \left[\frac{2}{s} \left(\frac{dB_1}{dT} \right) \mathbf{a}^\mu \omega_{\mu a} + \left(\frac{2B_1}{Ts} \right) \nabla_\mu \omega^{\mu a} \right] \left(\frac{\eta P_b^a \nabla_c \sigma^{cb} + \zeta P^{ca} \nabla_c \Theta}{s} \right) \\
&= \left[\frac{2}{s} \left(\frac{dB_1}{dT} \right) + \frac{2B_1}{Ts} \right] \mathbf{a}^\mu \omega_{\mu a} \left(\frac{\eta P_b^a \nabla_c \sigma^{cb} + \zeta P^{ca} \nabla_c \Theta}{s} \right) \\
&\quad - \frac{2B_1}{Ts} \left[P_{a\nu} \nabla_\mu \sigma^{\mu\nu} - \frac{2}{3} P_a^\mu \nabla_\mu \Theta + R_{\mu\nu} u^\nu P_a^\mu \right] \left(\frac{\eta P_b^a \nabla_c \sigma^{cb} + \zeta P^{ca} \nabla_c \Theta}{s} \right)
\end{aligned} \tag{B.5}$$

Divergence of the terms with coefficients B_2 and B_3 :

$$\begin{aligned}
& \nabla_\mu [B_2 \Theta^2 u^\mu]_{\text{upto 3rd order}} = \Theta^3 \left(B_2 - s \frac{dB_2}{ds} \right) + 2B_2 \Theta (u \cdot \nabla) \Theta \\
& \nabla_\mu [B_3 \sigma^2 u^\mu]_{\text{upto 3rd order}} = \sigma^2 \Theta \left(B_2 - s \frac{dB_2}{ds} \right) + 2B_3 \sigma_{\mu\nu} (u \cdot \nabla) \sigma^{\mu\nu}
\end{aligned} \tag{B.6}$$

Divergence of the term with coefficient B_4 :

$$\begin{aligned}
& \nabla_\mu (B_4 [(\nabla s)^2 u^\mu + 2s \Theta \nabla^\mu s])|_{\text{upto 3rd order}} \\
&= - \left(s \frac{dB_4}{ds} + B_4 \right) s^2 \Theta^3 - 2B_4 T^2 \left(\frac{ds}{dT} \right)^2 \mathbf{a}_\mu \mathbf{a}_\nu \sigma^{\mu\nu} + 2B_4 s \Theta \nabla^2 s \\
&\quad + \mathbf{a}^2 \Theta \left[s T^2 \left(\frac{ds}{dT} \right) \left(\frac{dB_4}{dT} \right) + \frac{T^2}{3} \left(\frac{ds}{dT} \right)^2 B_4 + 2B_4 T s \left(\frac{ds}{dT} \right) \right]
\end{aligned} \tag{B.7}$$

where $\nabla^2 s$ can be further simplified.

$$\begin{aligned}
\nabla^2 s &= -T \left(\frac{ds}{dT} \right) \frac{\Theta^2}{3} + \left(s - T \frac{ds}{dT} \right) (u \cdot \nabla) \Theta \\
&\quad + \left[T \frac{ds}{dT} + T^2 \frac{d^2 s}{dT^2} \right] \mathbf{a}^2 - T \frac{ds}{dT} [\sigma^2 + \omega^2 + R_{00}]
\end{aligned}$$

There is no relevant part in the fourth order corrections to the equations (B.6) and (B.7) (i.e. all the terms appearing in the fourth order corrections to these equations involve atleast one factor of Θ or $\sigma_{\mu\nu}$).

References

- [1] P. Romatschke, *Relativistic Viscous Fluid Dynamics and Non-Equilibrium Entropy*, *Class. Quant. Grav.* **27** (2010) 025006, [[arXiv:0906.4787](#)].
- [2] I. Kanitscheider and K. Skenderis, *Universal hydrodynamics of non-conformal branes*, *JHEP* **0904** (2009) 062, [[arXiv:0901.1487](#)].
- [3] E. M. Lifshitz and L. D. Landau, *Course of Theoretical Physics, volume 6*, .
- [4] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, *Relativistic viscous hydrodynamics, conformal invariance, and holography*, *JHEP* **0804** (2008) 100, [[arXiv:0712.2451](#)].
- [5] R. Loganayagam, *Entropy Current in Conformal Hydrodynamics*, *JHEP* **0805** (2008) 087, [[arXiv:0801.3701](#)].
- [6] K. Jensen *et. al.*, *Parity-Violating Hydrodynamics in 2+1 Dimensions*, [arXiv:1112.4498](#).
- [7] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom, *A Theory of first order dissipative superfluid dynamics*, [arXiv:1105.3733](#).
- [8] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, *et. al.*, *Local Fluid Dynamical Entropy from Gravity*, *JHEP* **0806** (2008) 055, [[arXiv:0803.2526](#)].